Inverse Limits and Models with Ill-Defined Forward Dynamics

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Abstract

Some economic models like the cash-in-advance model of money have the property that the dynamics are ill-defined going forward in time, but well-defined going backward in time. In this paper, we apply the theory of inverse limits to characterize topologically all possible solutions to a dynamic economic model with this property. We show that such techniques are particularly well-suited for analyzing the dynamics going forward in time even though the dynamics are ill-defined in this direction. In particular, we analyze the inverse limit of the cash-in-advance model of money and illustrate how information about the inverse limit is useful for detecting or ruling out complex dynamics.

Keywords: cash-in-advance, chaos, inverse limits, continuum theory.

JEL: C6, E3, and E4.

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1 Introduction

In this paper we apply the theory of inverse limits to characterize topologically equilibria in an economic model with ill-defined forward dynamics. The equilibrium of a dynamic economic model can often be characterized as a solution to a dynamical system. Many nonlinear dynamical systems are well-defined moving forward, but not well-defined going backward. However, in economics we sometimes have just the opposite, namely dynamics that are not well-defined going forward, but are well-defined going backward. Two such models include the overlapping generations (OLG) model and the cash-in-advance (CIA) model. Typically, the problem of ill-defined forward dynamics is either ignored by using a local analysis or avoided by analyzing the model with the well-defined backward map. However, using a local analysis ignores some potentially interesting equilibria. The backward map solution is unsatisfactory because the backward map gives orbits that go backward into the infinite past, whereas economists are interested in orbits that lead off into the infinite future.

Heuristically, the inverse limit of a dynamical system is a subset of an infinite dimensional space (e.g. the Hilbert cube) where each point in the inverse limit corresponds to a backward solution (backward orbit) of the dynamical system. Using the backward map from say the CIA model as our dynamical system, a point in the inverse limit, being a backward orbit of the backward map, corresponds to a forward orbit in the model. So we see that inverse limits are the right tool for analyzing models where the dynamics are well-defined going backward in time but not forward. For one-dimensional dynamical systems on an interval, the inverse limit has several important properties. First, the inverse limit is a continuum so the tools from topology that deal with continuum theory can be applied. Second, it is chainable and therefore can be topologically identified with an image in the plane.

Two questions naturally arise: (1) what does the structure of the inverse limit tell us about the underlying dynamics, and (2) what does the underlying dynamics imply for the inverse limit? Ingram and Mahavier (2004) explore the connections between these two aspects of dynamical systems for maps on the unit interval. Working within a family of one-dimensional maps, they illustrate that restrictions on parameters that lead to simple/complicated dynamics also lead to simple/complicated inverse limits. Barge and Martin (1985) show that if \( f : X \to X \) is a piecewise monotone map with \( X \) being an interval, then \( f \) has a periodic point of power not equal to a power of 2 if and only if \( \lim \left \langle I, f \right \rangle \) contains

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1 See Grandmont (1985) for the OLG model and Michener and Ravikumar (1998) for the CIA model.
2 Medio and Raines (2004a,b) also explore the applicability of inverse limits to economic models with ill-defined forward dynamics.
a chainable indecomposable continuum.\textsuperscript{3} Barge and Diamond (1994) extend this result for piecewise monotone map $f$ on a finite graph $X$. In this case, they show that the dynamical system will exhibit positive topological entropy if and only if the inverse limit contains an indecomposable continuum. Consequently, by analyzing the inverse limit one can detect chaotic/complex dynamics or rule out chaotic/complex behavior based on properties of the inverse limit. This offers an alternative method of detecting chaotic behavior that does not work directly with establishing a 3-cycle (or some other cycle and using Sarkovskii’s ordering on the integers) and ruling out chaotic behavior as well.

The paper is organized as follows. In section 2 we briefly cover the cash-in-advance model focusing on some of the properties of the implicitly-defined difference equation from that model. Next, we review some basic results from the research on inverse limits in section 3. Section 4 discusses why inverse limits are the right tool for analyzing models with ill-defined forward dynamics. In section 5, we illustrate how inverse limits can be used in the CIA model to understand the possible dynamics in the ill-defined direction. We conclude in section 6.

2 The Model

In this section, we describe the cash-in-advance model with particular emphasis on the problem of ill-defined forward dynamics. The model is the standard endowment CIA model of Lucas and Stokey (1987). We closely follow the exposition of Michener and Ravikumar (1998), hereafter [MR]. It is an endowment economy with both cash and credit goods. There is a representative agent and a government. The government consumes nothing and sets monetary policy using a money growth rule.

The household has preferences over sequences of the cash good ($c_{1t}$) and credit good ($c_{2t}$) represented by a utility function of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}),$$

with the discount factor $0 < \beta < 1$.

To purchase the cash good $c_{1t}$ at time $t$ the household must have cash $m_t$. This cash is carried forward from $t - 1$ and in this sense the household is required to have cash in advance of purchasing the cash good. The credit good $c_{2t}$ does not require cash, but can be bought on credit. The household has an endowment $y$ each period that can be transformed into the cash and credit goods according to $c_{1t} + c_{2t} = y$. Since this technology allows the cash good

\textsuperscript{3}These objects are defined and discussed in section 3. For now, we note that chainable indecomposable continua are topologically equivalent to certain fractal objects in the plane.
to be substituted for the credit good one-for-one, both goods must sell for the same price \( p_t \) in equilibrium and the endowment must be worth this price per unit as well. Each period the household also receives a transfer of cash from the government in the amount \( \theta M_t \).

The household seeks to maximize (1) by choice of \( \{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty} \) subject to the constraints \( c_{1t}, c_{2t}, m_{t+1} \geq 0 \),

\[
\begin{align*}
p_t c_{1t} &\leq m_t, \\
m_{t+1} &\leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t},
\end{align*}
\]

(2) taking as given \( m_0 \) and \( \{p_t, M_t\}_{t=0}^{\infty} \). The money supply \( \{M_t\} \) is controlled by the government and follows a constant growth path \( M_{t+1} = (1+\theta)M_t \) where \( \theta \) is the growth rate and \( M_0 > 0 \) given.

Equation (2) is the cash-in-advance constraint which says that the amount spent on the cash good \( p_t c_{1t} \) must be no more than cash on hand \( m_t \). Equation (3) is the budget constraint on cash holdings for next period. In words, it says that the cash carried over into next period \( (m_{t+1}) \) can be no greater than the income \( (p_t y) \) plus cash not spent \( (m_t - p_t c_{1t}) \) plus the transfer of cash from the government \( (\theta M_t) \) minus the amount spent on the credit good \( (p_t c_{2t}) \). [MR] make assumptions on the function \( U \) so that the solution to this problem will be interior and the solution to the first-order conditions and transversality condition will be necessary and sufficient.

Assumption 1. ([MR], p. 1120) The function \( U : R^2_+ \rightarrow R \) is \( C^2 \) with \( U_1 > 0, U_2 > 0 \) and the Hessian matrix negative definite. Both \( c_{1t} \) and \( c_{2t} \) are assumed to be normal goods. Further, to guarantee interior solutions we will assume

\[
\lim_{c \to 0} U_1(c, y - c) = \lim_{c \to y} U_2(c, y - c) = \infty,
\]

and that \( U_1(y, 0) < \infty \) and \( U_2(0, y) < \infty \).

To solve the household’s constrained optimization problem, we set up the Lagrangian:

\[
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{U(c_{1t}, c_{2t}) + \mu_t (m_t - p_t c_{1t}) + \lambda_t (p_t(y - c_{2t} - c_{1t}) + m_t - m_{t+1} + \theta M_t)\}
\]

where \( \{\mu_t, \lambda_t\} \) are non-negative Lagrange multipliers. The first-order conditions from this problem imply that

\[
U_2(c_{1t}, c_{2t})/p_t = \beta U_1(c_{1t+1}, c_{2t+1})/p_{t+1}.
\]

(4) This condition reflects that at the optimum, the household must be indifferent between spending a little more on the credit good (giving a marginal benefit \( U_2(c_{1t}, c_{2t})/p_t \)) versus
savings the money and purchasing the cash good in the next period (giving a marginal benefit
\[ \beta U_2(c_{1t+1}, c_{2t+1})/p_{t+1} \].

A perfect foresight equilibrium is defined in the usual way as a collection of sequences
\( \{c_{1t}, c_{2t}, m_t\}_{t=0}^{\infty} \) and \( \{M_t, p_t\}_{t=0}^{\infty} \) satisfying the following.

(1) The money supply follows the stated policy rule: \( M_{t+1} = (1 + \theta)M_t \).

(2) Markets clear: \( m_t = M_t \) and \( c_{1t} + c_{2t} = y \).

(3) The solution to the household optimization problem is given by \( \{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty} \).

Let \( x_t := m_t/p_t \) denote the level of real money balances. Using the equilibrium conditions
that \( M_t = m_t \) and \( c_{2t} = y - c_{1t} \), equation (4) implies
\[ x_t U_2(c_{1t}, y - c_{1t}) = \frac{\beta}{1 + \theta} x_{t+1} U_1(c_{1t+1}, y - c_{1t+1}). \] (5)

If the cash-in-advance constraint (2) binds, then \( c_{1t} = x_t \). If not, then the Lagrange multiplier
\( \mu_t = 0 \) and \( c_{1t} = c := \arg \max_{x} U(x, y - x) \). It then follows that \( c_{1t} = \min\{x_t, c\} \) for all \( t \).
Using this relationship we can eliminate \( c_{1t} \) and \( c_{1t+1} \) from (5) to get a difference equation
in \( x \) alone:
\[ x_t U_2(\min\{x_t, c\}, y - \min\{x_t, c\}) = \frac{\beta}{1 + \theta} x_{t+1} U_1(\min\{x_{t+1}, c\}, y - \min\{x_{t+1}, c\}) \]
or
\[ B(x_t) = A(x_{t+1}), \] (6)
where
\[ B(x) := x U_2(\min\{x, c\}, y - \min\{x, c\}), \]
\[ A(x) := \frac{\beta}{1 + \theta} x U_1(\min\{x, c\}, y - \min\{x, c\}). \]

Whether or not the dynamics going forward are well-defined depends on whether or not \( A(\cdot) \)
is invertible. See Figure 1 for illustrations of two possible configurations for \( A \) and \( B \).

[MR] use two more assumptions in their paper which we include here for completion and
briefly describe what they imply for the model.

**Assumption 2.** ([MR], p. 1125) There exists a \( b \in [0, c) \) such that \( x U_1(x, y - x) \) is increasing
in the region \([0, b)\) and decreasing in the region \((b, c]\).

This assumption is putting additional restrictions on the utility function so that the
function \( A(\cdot) \) is either hump-shaped or monotonically decreasing on \([0, c]\).
Assumption 3. ([MR], p. 1125) (a) \((1 + \theta) > \beta\) and (b) \(b < x^*\).

These conditions guarantee the existence of a solution \(x^* > 0\) to \(A(x^*) = B(x^*)\) and that this intersection of the two functions occurs when \(A(x)\) is decreasing.

One can show that there is a one-to-one mapping between equilibria in the model and non-negative sequences \(\{x_t\}\) that satisfy the difference equation (6) and transversality condition

\[
\lim_{t \to \infty} \beta^t U_1(\min[x_t, c], y - \min[x_t, c]) x_t = 0.
\]

Since the discount factor \(\beta\) is assumed to be strictly between 0 and 1, any solution to the difference equation (6) that is bounded from above and from below by a strictly positive constant will satisfy the transversality condition. Consequently, in the discussion that follows, solutions to (6) that satisfy \(0 < x_l < x_t < x_u < \infty\) for all \(t\) will be an equilibrium in the model.

### 2.1 Properties of the Implicit Difference Equation

Consider the difference equation defined by (6) from above and recall that we are interested in the solutions to the difference equation, which are sequences \(x_0, x_1, x_2, \ldots\) of nonnegative real numbers satisfying the difference equation. Under assumptions 1–3, the functions \(A\) and \(B\) are continuous functions from \([0, \infty)\) to \([0, \infty)\) with the following properties:

1. \(B\) is increasing and therefore one-to-one, but \(A\) is not one-to-one.

2. For some positive number \(c\), both \(A\) and \(B\) are linear on \([c, \infty)\) with positive slopes, and the slope of \(A|[c, \infty)\) is less than the slope of \(B|[c, \infty)\).

3. On some interval \([0, b]\), (with \(b < c\)) the behavior of \(A\) may be increasing with \(A(0) = 0\) (type I), or it may be increasing with \(A(0) > 0\) (type II), or \(A\) may be decreasing on \([0, c]\) (type III). For type III we let \(b = 0\).

4. On the interval \((b, c]\), \(A\) is decreasing, with \(x \in (b, c]\) such that \(A(x) = B(x)\).

Note that there are positive numbers \(\underline{x}\) and \(\overline{x}\) such that

\[
B(\underline{x}) = A(c),
B(\overline{x}) = A(x),
\]

and in type I and possibly type II there are positive numbers \(\underline{x}^b\) and \(\overline{x}^b\) such that

\[
B(\underline{x}^b) = A(b),
B(\overline{x}^b) = A(\overline{x}).
\]
Since the function $A$ is not one-to-one, the dynamics in the model given by the difference equation (6) are not well-defined going forward in time. However, since $B$ is one-to-one, we can invert $B$ and define the backward map $f(x) := B^{-1} \circ A(x)$. This function gives the backward dynamics $x_t = f(x_{t+1})$, maps $[0, \infty)$ to itself and inherits the basic shape of $A$. Consequently, even though the dynamics of (6) are not well-defined going forward in time, the dynamics are well-defined going backward in time. In terms of the $f$ function we have:

- $x = f(c)$,
- $\bar{x} = f(x)$,
- $\bar{x}^b = f(\bar{x}^b)$.

Note that if $b > 0$, we have $A(x) \leq A(b)$, which implies $B(x) = A(x) \leq A(b) = B(\bar{x}^b)$. Since $B$ is increasing, this implies $\bar{x} \leq \bar{x}^b$. Also, if $b > 0$, we have $A(\bar{x}^b) \geq A(c)$ (since $\bar{x}^b > b$), which implies $B(x) = A(c) \leq A(\bar{x}^b) = B(\bar{x}^b)$. Since $B$ is increasing, this implies $x \leq \bar{x}^b$. [MR] provide the following propositions about the attracting sets for the backward map $f$.

**Proposition 1** ([MR], Lemma 2, p. 1125). If $0 < b \leq \bar{x}$, then the attracting set for $f$ is $J := [x, \bar{x}]$, i.e., for any solution $\ldots, x_{-2}, x_{-1}, x_0$ with $x_0 > 0$ to $x_{t-1} = f(x_t)$ there exists some negative integer $T$ such that $x_t \in J$ for all $t \leq T$.

With $0 < b \leq \bar{x}$ and $J := [x, \bar{x}]$, there are three generic possibilities for $f|J$ (see Figure 2):

- **I.A.** $\bar{x} > c$,
- **I.B.** $\bar{x} = c$,
- **I.C.** $\bar{x} < c$. 

Figure 1: Type I (left) and III (right) configurations for $A$ and $B$. 

- **Figure 1**: Type I (left) and III (right) configurations for $A$ and $B$. 

- **Figure 2**: Diagram showing the backward dynamics and attracting sets.
Figure 2: The figure illustrates the generic possible shapes for \( f : J \rightarrow J \) in each of cases I.A–I.C when \( 0 < b \leq x \).

![Figure 2](image)

Proposition 2 ([MR], Lemmas 3 and 4, pp. 1126–27). Under assumptions 1–3, if \( x < b \) the attracting set for \( f \) depends on the relative magnitudes of \( c \) and \( x \). If \( x \leq c \), then the attracting set for \( f \) is \( J := [x^b, x^b] \), i.e., for any solution \( \ldots, x_{-2}, x_{-1}, x_0 \) with \( x_0 > 0 \) to \( x_{t-1} = f(x_t) \) there exists some negative integer \( T \) such that \( x_t \in J \) for all \( t \leq T \). If \( x > c \), then the attracting set for \( f \) is \( J := [x, x^b] \), i.e., for any solution \( \ldots, x_{-2}, x_{-1}, x_0 \) with \( x_0 > 0 \) to \( x_{t-1} = f(x_t) \) there exist some negative integer \( T \) such that \( x_t \in J \) for all \( t \leq T \).

With \( b > x \), there are four generic possibilities for \( f|_J \) (see Figure 3):

II.A. \( x \leq x^b < b < x^b \leq c \) with \( J := [x^b, x^b] \),

II.B. \( x < b = x^b < x^b \leq c \) with \( J := [x^b, x^b] \),

II.C. \( x \leq x^b < b < x^b \leq c \), with \( J := [x^b, x^b] \),

II.D. \( x < b < c < x^b \) with \( J := [x, x^b] \).

Figure 3: The figure illustrates the generic possible states for \( f|_J : J \rightarrow J \) in each of the cases II.A–II.D when \( x < b \).

![Figure 3](image)

In this paper, we only consider the case where \( b > 0 \) with \( A(0) = 0 \) (type I).\(^4\) In particular we are going to look at cases I.A, I.B, I.C, II.B, and II.C.\(^5\) We leave cases II.A and II.D for future research.

\(^4\)Type II with \( A(0) > 0 \) is essentially the same, but one has to be more careful because solutions for \( x_{t+1} \) to \( A(x_{t+1}) = B(x_t) \) may not exists for low values of \( x_t \).

\(^5\)In a companion paper for a math journal, Kennedy et al. (2004) explore the inverse limit for case I.A only.
Consider case I.A. Note that if a sufficiently large \( x_0 \) is chosen, then the requirement that \( A(x_1) = B(x_0) \) forces \( x_1 \) to be larger than \( x_0 \), and \( x_1 \) is unique (see Figure 1.) Continuing, one sees that the solution \((x_0, x_1, \ldots)\) for that initial condition is well-defined and \( \lim_{t \to \infty} x_t = \infty \). Likewise, if a sufficiently small positive \( x_0 \) is chosen, then \( x_1 \) is smaller than \( x_0 \), and the solution \((x_0, x_1, \ldots)\) consists of a decreasing sequence of positive numbers converging to 0.

We summarize the possibilities precisely in the following propositions. Note that the five propositions below deal with forward solutions to the difference equation (6), whereas the previous two propositions from [MR] deal with backward solutions.

**Proposition 3.** For cases I.A–I.C and II.D, if \((x_0, x_1, \ldots)\) is a solution to \( A(x_{t+1}) = B(x_t) \) such that \( x_{\hat{t}} < \bar{x} \) for some \( \hat{t} \), then

[a] for \( t \geq \hat{t} \), the choice of \( x_{t+1} \) is unique, i.e., \( x_{t+1} \) such that \( A(x_{t+1}) = B(x_t) \) is unique;

[b] \( \lim_{t \to \infty} x_t = 0 \); and

[c] \( x_{\hat{t}} > x_{\hat{t}+1} > x_{\hat{t}+2} > \cdots \).

**Proposition 4.** For case I.A–I.C, if \((x_0, x_1, \ldots)\) is a solution to \( A(x_{t+1}) = B(x_t) \) such that \( x_{\hat{t}} > \bar{x} \) for some \( \hat{t} \), then the choice of \( x_{\hat{t}+1} \) may not be unique, but either

[a] \( \lim_{t \to \infty} x_t = \infty \) and eventually \( x_t < x_{t+1} < x_{t+2} < \cdots \), or

[b] \( \lim_{t \to \infty} x_t = 0 \) and eventually \( x_t > x_{t+1} > x_{t+2} > \cdots \).

**Proposition 5.** For cases II.A–II.D, if \((x_0, x_1, \ldots)\) is a solution to \( A(x_{t+1}) = B(x_t) \) such that \( x_{\hat{t}} > \bar{x}^b \) for some \( \hat{t} \), then

[a] for \( t \geq \hat{t} \), the choice of \( x_{t+1} \) is unique, i.e., \( x_{t+1} \) such that \( A(x_{t+1}) = B(x_t) \) is unique;

[b] \( \lim_{t \to \infty} x_t = \infty \); and

[c] \( x_{\hat{t}} < x_{\hat{t}+1} < x_{\hat{t}+2} > \cdots \).

**Proposition 6.** For case II.A–II.C, if \((x_0, x_1, \ldots)\) is a solution to \( A(x_{t+1}) = B(x_t) \) such that \( x_{\hat{t}} < \bar{x}^b \) for some \( \hat{t} \), then the choice of \( x_{\hat{t}+1} \) may not be unique, but either

[a] \( \lim_{t \to \infty} x_t = \infty \) and eventually \( x_t < x_{t+1} < x_{t+2} < \cdots \), or

[b] \( \lim_{t \to \infty} x_t = 0 \) and eventually \( x_t > x_{t+1} > x_{t+2} > \cdots \).
In summary, Propositions 3 and 4 imply that for cases I.A–I.C, any forward solution that leaves \([x, \overline{x}]\) is not interesting. Propositions 3 and 5 imply that for case II.D, any forward solution that leaves \([x, x_b]\) is not interesting. And finally, Propositions 5 and 6 imply that for cases II.A–II.C, any forward solution that leaves \([x_b, x_b]\) is not interesting.

For cases I.A, I.B, II.A, II.B, and II.D, \(f|J : J \rightarrow J\) is onto, so if \(x_t \in J\) there is a point \(x_{t+1} \in J\) such that \(x_t = f(x_{t+1})\). However, in in cases I.C and II.C, the map \(f|J : J \rightarrow J\) is not onto, this implies that moving forward in time, some points must be thrown out of \(J\) implying uninteresting dynamics. Since we are interested in potentially interesting dynamics, we remove these points from \(J\). Let \(K\) be the the collection of points not removed (this is nonempty since the steady state solution \(x^* \in J\)). Since \(f|J\) is monotonic, one can show that \(K\) must be of the form \(K := [z, \overline{z}] \subset J\) with \(z \leq \overline{z}\). Note that that \(f|K : K \rightarrow K\) is onto and the picture looks similar to I.B or II.B if \(z < \overline{z}\). However, it is possible for \(z = \overline{z}\), in this case the only bounded solution is \(x_t = x^*\). To simplify notation, we will denote \(K\) by \(J\) and always consider \(f|J : J \rightarrow J\).

It follows from the previous propositions that in cases I.A–I.C and II.A–II.D solutions that contain members not in the interval \(J\) exhibit simple behavior. From a dynamics perspective, they are not very interesting. From an economics perspective, they may not constitute an equilibrium (the transversality condition may be violated). If the transversality condition is satisfied in these cases, then such equilibria are referred to as self-fulfilling inflations \((x_t \rightarrow 0)\) and self-fulfilling deflations \((x_t \rightarrow \infty)\). Moreover, a solution containing a member not in \(J\) would be locked into one behavior - either its members would eventually increase without bound, or they would eventually decrease to 0.

We close this section with a characterization theorem for type I.A backward maps in the CIA model. By type I.A, we mean \(f : [x, \overline{x}] \rightarrow [x, \overline{x}]\) with \(0 < \underline{x} < \overline{x}\), \(f(\overline{c}) = \overline{x}\) for some \(\overline{x} < \overline{c} < \overline{x}\) with \(f(\overline{x}) = \overline{x}\), \(f|[x, \overline{c}]\) one-to-one and \(f|[\overline{c}, \overline{x}]\) one-to-one.

**Theorem 1.** In the CIA model, \(f\) is a type I.A backward map generated by a utility function \(U(c_1, c_2)\) satisfying Assumption 1 with \(0 < \tilde{\beta} < 1\) if and only if \(f\) is a type I.A map satisfying

1. for \(x \geq x_2\), \(f\) is linear with slope \(0 < \tilde{\beta} < 1\) with \(x_2 = \bar{c}/\tilde{\beta}\),
2. \(f\) is \(C^1\) on \([x, \overline{x}] \setminus \{x_1, \overline{c}, x_2\}\), where \(\underline{x} < x_1 < \overline{c}\) is the unique solution to \(f(x) = \overline{c}\),
3. for \(x \in \{x_1, \overline{c}, x_2\}\), \(f'(x^+)\) and \(f'(x^-)\) exist, are non-zero with

\[
\frac{f'(x^-)}{f'(x^+)} = \frac{f'(x_2^+)}{f'(x_2^-)}
\]

and

\textsuperscript{6}See Woodford (1994) for a careful discussion of these cases.
4. \( f'(x)x < f(x) \) for \( x \in [\bar{c}, x_2] \).

Proof. (\( \Rightarrow \)): Let \( f : [\underline{x}, \bar{x}] \to [\underline{x}, \bar{x}] \) given by \( f(x) := B^{-1}(A(x)) \) be a type I.A backward map. We have \( \bar{x} = f(\bar{c}) \), \( f(\bar{c}) = \bar{c} \) with \( \underline{x} < \bar{c} < \bar{x} \). \( f \) is continuous since \( A \) and \( B^{-1} \) are continuous. \( f|_{[\underline{x}, \bar{c}]} \) one-to-one, and \( f|_{[\bar{c}, \bar{x}]} \) one-to-one. Recall we have
\[
B(x) := xU_2(\min[x, \bar{c}], y - \min[\bar{c}, x]), \\
A(x) := \bar{\beta}xU_1(\min[x, \bar{c}], y - \min[\bar{c}, x]).
\]

Note there exists a unique \( \underline{x} < x_1 < \bar{c} \) such that \( f(x_1) = \bar{c} \). Let \( x_2 = \bar{c}/\bar{\beta} \). Then for \( x \geq x_2 \), \( f(x) = \bar{\beta}x \). We have \( x_2 < \bar{x} \) iff \( f(\bar{x}) > \bar{c} \). For \( [\underline{x}, \bar{x}] \setminus \{x_1, \bar{c}, x_2\} \), \( f \) is at least \( C^1 \). These points where \( f \) may fail to be \( C^1 \) occur when \( A(x) = \bar{c} \) through the kink in \( B(\cdot) \) and through the kink in \( A(\cdot) \) at \( \bar{c} \). We consider the case where \( x_2 < \underline{x} \) (the case with \( x_2 \geq \bar{x} \) is similar).

By the properties of \( A \) and \( B \), it is easily verified that \( f' \) has right and left limits (all non-zero) at \( \{x_1, \bar{c}, x_2\} \) given by
\[
f'(x_1) = A'(x_1)/B'(\bar{c}^+), \quad f'(x_1^+) = A'(x_1)/B'(\bar{c}^-), \\
f'(\bar{c}^-) = A'(\bar{c}^-)/B'(\bar{x}), \quad f'(\bar{c}^+) = A'(\bar{c}^-)/B'(\bar{x}), \\
f'(x_2) = A'(x_2)/B'(\bar{c}^-), \quad f'(x_2^+) = A'(x_2)/B'(\bar{c}^-).
\]

Taking ratios shows that
\[
\frac{f'(x_1^-)}{f'(x_1^+)} = \frac{A'(x_1)/B'(\bar{c}^+)}{A'(x_1)/B'(\bar{c}^-)} = \frac{A'(x_2)/B'(\bar{c}^-)}{A'(x_2)/B'(\bar{c}^-)} = \frac{f'(x_2^+)}{f'(x_2^+)}. \\
\]

For \( \bar{c} \leq x \leq x_2 \), we have \( A(x) = \bar{\beta}U_1x \) so \( B(f(x)) = A(x) = \bar{\beta}U_1x \) or
\[
f(x)U_2(f(x), y - f(x)) = \bar{\beta}U_1x.
\]
This implies
\[
U_2(f(x), y - f(x)) \equiv \bar{\beta}U_1x/f(x) > 0,
\]
and
\[
U_{22}(f(x), y - f(x)) - U_{21}(f(x), y - f(x)) = \bar{\beta}U_1 \left[ \frac{f'(x)x - f(x)}{f'(x)x^2} \right].
\]
Since \( D^2U \) is negative definite and both the cash and credit goods being normal goods, we have \( U_{22}(f(x), y - f(x)) - U_{21}(f(x), y - f(x)) < 0 \) implying that we must have \( f'(x)x < f(x) \) for \( \bar{c} \leq x \leq x_2 \).

(\( \Leftarrow \)): Let \( f : [\underline{x}, \bar{x}] \) be a type I.A map satisfying the hypotheses of the theorem. Our goal is construct a utility function satisfying Assumption 1 that generates \( f \). Since \( f \) maps \([\underline{x}, \bar{x}]\) into itself, we can use \( f \) to generate a utility function \( U : [\underline{x}, \bar{c}] \times [y - \bar{c}, y - \underline{x}] \to \mathbb{R} \).
satisfying Assumption 1 and then extend the domain in a way consistent with Assumption 1. We are going to do this with a separable utility function $U(c_1, c_2) = u(c_1) + w(c_2)$. As a normalization, we can let $1 = \bar{U}_1 := u(\bar{c})$ and $1 = \bar{U}_2 = w(y - \bar{c})$. Let $I_1 := [x, x_1], I_2 := [x_1, \bar{c}], I_3 := [\bar{c}, x_2]$, and $I_4 := [x_2, \bar{x}]$.

On $I_3 \cup I_4$, we define $B(x) := x\bar{U}_2$ and $A(x) := \beta x\bar{U}_1$. These $A$ and $B$ will generate $f$ on $I_4$. To see this, for $x \in I_4$, we have $f(x) \geq \bar{c}$ and $f(x) = \beta x$. So $B(f(x)) = f(x)\bar{U}_2 = \beta x\bar{U}_1 = A(x)$, since $\bar{U}_1 = \bar{U}_2$.

On $I_3$ we have $f$ one-to-one with $f(I_3) = [x, \bar{c}]$. Let $\hat{f} := (f|I_3)^{-1}$. Define $w'(\cdot)$ on $[y - \bar{c}, y - x]$ from

$$w'(y - f(x)) = \beta \bar{U}_1 x/f(x).$$

for $x \in I_3$. We clearly have

$$w'(y - f(x)) = \beta \bar{U}_1 x/f(x) > 0,$$

and

$$w''(y - f(x))( - f'(x)) = \beta \bar{U}_1 \left[ \frac{f(x) - f'(x)x}{f(x)^2} \right] < 0$$

since $f(x) > f'(x)x$ for $x \in I_3$. $w''$ is continuous since $f$ is $C^1$ on $I_3$. To recover $w(\cdot)$, note that for $x \in [y - \bar{c}, y - x]$, we have

$$w'(x) = \beta \bar{U}_1 \frac{\hat{f}(y - x)}{y - x}.$$ 

Then for $x \in [y - \bar{c}, y - x]$ we have

$$w(x) := C_w + \int_x^y w'(z)dz.$$

Also for $x \in [\bar{x}, \bar{c}]$ define $B(x) := xu'(y - x)$, then by construction we have $B(f(x)) \equiv A(x)$ for $x \in I_3$. Note that we now have $B(x)$ defined on $[\bar{x}, \bar{c}]$.

On $I_1 \cup I_2$, define $A(x) := B(f(x))$ and define $u'(\cdot)$ on $I_1 \cup I_2$ by $u'(x) := A(x)/(\beta x) > 0$. Then

$$u(x) := C_u + \int_x^\bar{x} u'(z)dz.$$

Note that $A$ is $C^1$ on $I_1$ and on $I_2$ with $A'(x) = B'(f(x))f'(x) < 0$. Consequently $u'(\cdot)$ is $C^1$ on $I_1$ and on $I_2$ with $u''(x) = [A'(x)\beta x - \beta A'(x)]/(\beta x)^2 < 0$. For $u''$ to be continuous we need $A'$ to be continuous so we need to check whether or not $A'(x^-_1) = A'(x^+_1)$. By construction, we have $B'(\bar{c}^+)B'(\bar{c}^-) = f'(x^+_2)/f'(x^+_2)$. At $x_1$, we have $A'(x^-_1) = A'(x^+_1) = B'(f(x^-_1))f'(x^-_1) = B'(\bar{c}^+)f'(x^-_1)$ and $A'(x^+_1) = B'(f(x^+_1))f'(x^+_1) = B'(\bar{c}^-)f'(x^+_1)$. Taking ratios we have

$$\frac{A'(x^-_1)}{A'(x^+_1)} = \frac{B'(\bar{c}^+)f'(x^-_1)}{B'(\bar{c}^-)f'(x^+_1)} = 1$$

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since by the hypothesis of the theorem we have \( f'(x_1^+) / f'(x_1^-) = f'(x_2^-) / f'(x_2^+) \). So \( u'' \) is continuous on \( I_1 \cup I_2 \). By construction \( B(f(x)) \equiv A(x) \) on \( I_1 \cup I_2 \).

So \( u : [x, c] \to \mathbb{R} \) with \( u' > 0 \) and \( u'' < 0 \) and \( w : [y - c, y - x] \to \mathbb{R} \) with \( w' > 0 \) and \( w'' < 0 \). Both of these functions can be extended to \( \mathbb{R}_+ \), call the extensions \( \tilde{u} \) and \( \tilde{w} \) so that \( \tilde{U}(c_1, c_2) := \tilde{u}(c_1) + \tilde{w}(c_2) \) satisfies Assumption 1.

\[ \square \]

3 Dynamics, Continuum Theory, and Inverse Limits

The applicability of inverse limits to the CIA model depends on the relationship between equilibria in the model, the inverse limit, continuum theory, and dynamics. In this section we discuss the necessary background from dynamics, inverse limits and continuum theory for our application to the CIA model. We do not express definitions or theorems in their most general form, but at a level sufficient for our purposes.

3.1 Dynamics

Suppose \( X \) is a compact metric space and \( f : X \to X \) is continuous. The point \( x \) in \( X \) is a periodic point of period \( n \) if \( f^n(x) = x \) and \( n \) is the smallest positive integer with \( f^n(x) = x \). If \( x \) is a periodic point of period \( n \), then \( O_+(x) := \{ x, f(x), f^2(x), \ldots, f^{n-1}(x) \} \) is the orbit of \( x \). The orbit \( O_+(x) \) is attracting if there is an open set \( o \) in \( X \) containing \( O_+(x) \) such that if \( y \in o \), then \( \lim_{m \to \infty} f^{mn}(y) \) is some member of \( O_+(x) \). The basin of attraction of \( O_+(x) \) is the set \( \{ y \in X : \lim_{m \to \infty} f^{mn}(y) \text{ is some member of } O_+(x) \} \). (Thus, the basin of attraction for an attracting orbit is an open set that contains all points attracted to the orbit.) The orbit \( O_+(x) \) is repelling if there is an open set \( o \) in \( X \) containing \( O_+(x) \) such that if \( y \in o \), \( y \notin O_+(x) \), then there is some positive integer \( N_y \) such that if \( k > N_y \), then \( f^k(y) \notin o \). The basin of repulsion of \( O_+(x) \) is the set \( \cup\{ u : u \text{ is open in } X \text{ and for each } y \in u \backslash O_+(x), \text{ there is some positive integer } N_y \text{ such that if } k > N_y, f^k(y) \notin u \} \). (Thus, the basin of repulsion for a repelling orbit is an open set that contains all points pulled away from that orbit.)

We can talk about the orbits of points that are not periodic, too: If \( X \) is a metric space and \( f : X \to X \) is continuous, the orbit \( O_+(x) \) of the point \( x \) under the action of \( f \) is the set \( O_+(x) = \{ x, f(x), f^2(x), \ldots \} \). A subset \( A \) of \( X \) is invariant under \( f \) if \( f(A) = A \). (Hence the orbit of a periodic point is an invariant subset of \( X \) under \( f \).)

Suppose that \( X \) and \( Y \) are metric spaces, \( f : X \to X \) is continuous and \( g : Y \to Y \) is continuous. If there is a homeomorphism \( h : X \to Y \) such that \( h \circ f = g \circ h \), then \( f \) and \( g \) are said to be conjugate. Whenever two maps are conjugate, their dynamics are equivalent. For example, consider two maps on \([0, 1]\) into itself given by \( g(x) := 4x(1 - x) \) (the logistic map)
and \( f(x) := 1 - |2x - 1| \) (the tent map). One can show that these two maps are conjugate, where the conjugacy is given by \( h(x) := \left( \frac{1}{2} \right)[1 - \cos(\pi x)] \).

A subset \( A \) of a complete, separable metric space \( X \) is residual in \( X \) if \( A \) contains a dense \( G_\delta \) subset of \( X \).\(^7\) In a topological sense, dense \( G_\delta \) subsets of a complete separable metric space \( X \) are large in the space: they represent a set that is “almost all” of the space. Suppose that \( K \) is a compact, metric space, and \( h : K \to K \) is continuous. If there is a point \( p \) which has a dense orbit in \( K \) under the action of \( h \), then there is a residual set of points in \( K \) each of which has its orbit dense in \( K \). We say that \( h \) is transitive if there is a point \( p \) in \( K \) which has its orbit dense in \( K \). The map \( h \) is transitive if and only if it has the following property: if \( u \) and \( v \) are nonempty open subsets of \( K \), then there is some integer \( n \) such that \( f^n(u) \cap v \neq \emptyset \). The map \( h \) has sensitive dependence on initial conditions on the invariant closed subset \( H \) of \( K \) if there is some positive number \( r \) such that for each point \( x \) in \( H \) and for each \( \epsilon > 0 \), there is a point \( y \) in \( H \) with \( d(x, y) < \epsilon \) and an integer \( k \geq 0 \) such that \( d(h^k(x), h^k(y)) \geq r \). The map \( h \) is chaotic in the sense of Li and Yorke (1975) if \( h \) has sensitive dependence on initial conditions on \( K \). The map \( h \) is chaotic in the sense of Devaney (2003) if (1) there is a point \( p \) in \( K \) which has its orbit dense in \( K \), (2) the set of periodic points in \( K \) is dense in \( K \), and (3) \( h \) is sensitive to initial conditions at each point of \( K \).

If \( X \) is a metric space and \( f : X \to X \) is continuous, \( f \) is transitive, and the set of periodic points of \( f \) is dense in \( X \), then \( f \) has sensitive dependence on initial conditions [see Banks et al. (1992)]. Thus, Devaney’s last condition is redundant. Roe (1993) shows that if \( X \) is a finite tree and \( f : X \to X \) is continuous and has a dense orbit, then \( f \) is chaotic in the sense of Devaney.\(^8\) Thus, for a map from an interval (the simplest tree) to itself, condition (1) above implies that the map is chaotic in the sense of Devaney.

A map on an interval onto itself is called Markov if there is a finite invariant set \( A \) containing the end points of the interval such that if \( p \) and \( q \) are consecutive members of \( A \), then the restriction of the map to \([p, q]\) is monotone.

### 3.2 Inverse Limits and Continuum Theory

In this subsection, we discuss the concept of a continuum along with a few of its topological properties. We also introduce the notion of an inverse limit and discuss some relevant

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\(^7\)A \( G_\delta \) set is the intersection of a countable family of open sets. A simple example of a dense \( G_\delta \) subset of \([0, 1]\) is the set of irrationals in \([0, 1]\). Suppose that \( \{d_1, d_2, \ldots\} \) is a sequence that consists of the rational numbers in \([0, 1]\). For each positive integer \( i \) let \( u_i = [0, 1] - \{d_i\} \) so that \( u_i \) is open and dense in \([0, 1]\). Let \( G = \cap_{i=1}^{\infty} u_i \) so that \( G \) is the intersection of dense open sets in \([0, 1]\), i.e., a \( G_\delta \) set. Then \( G \) is precisely the set of irrational numbers in \([0, 1]\).

\(^8\)Note that Roe’s assumption that \( X \) be a tree is important: An irrational rotation on a circle forms a dynamical system in which every orbit is dense, but it is not chaotic and it does not have any periodic points.
theorems for our application to the CIA model.

**Definition 1.** A space is connected if and only if it is not the union of two disjoint, closed and nonempty sets.

**Definition 2.** A continuum is a compact, connected metric space. If $X$ and $Y$ are continua, and $Y \subset X$, then $Y$ is a subcontinuum of $X$. If $Y$ is a subcontinuum of $X$, but $Y \neq X$, then $Y$ is a proper subcontinuum of $X$.

**Definition 3.** An arc is a space homeomorphic to the unit interval $[0,1]$. An arc continuum is a continuum with the property that each of its proper subcontinua is homeomorphic to an arc.

One simple example of a continuum is the unit interval $I := [0,1]$. A subcontinuum of $I$ is $[1/2,3/4]$. An arc is of course an arc continuum. A more interesting example of an arc continuum is the Knaster bucket handle (discussed below). Two properties of continua that are relevant to this paper are the concepts of being chainable and indecomposable.

**Definition 4.** A chain is a finite sequence $G_1, G_2, \ldots, G_n$ of open sets such that $G_i$ intersects $G_j$ if and only if $|i - j| \leq 1$. The open sets $G_i$ are the links of the chain. The mesh of the chain is the largest diameter of its links. A continuum is chainable provided for each $\epsilon > 0$ there is a chain cover of $M$ with mesh less than $\epsilon$.

**Definition 5.** A continuum is decomposable if it is the union of two of its proper subcontinua, otherwise it is indecomposable.

Again, the unit interval $I$ is a simple example of a continuum that is both chainable and decomposable. For example, $I = [0,1/2] \cup [1/4,1]$, and so $I$ is decomposable. For $\epsilon > 0$, let $N$ be a natural number such that $1/N < \epsilon$. Let $G_i = (a_i - 3/4N, a_i + 1/2N)$ with $a_i = i/N$ for $i = 0, 1, \ldots, N$. The collection $\{G_i\}_{i=1}^{\infty}$ is a chain cover of $I$ with mesh less than $\epsilon$, so $I$ is chainable.

An indecomposable continuum is central to our investigation of the dynamics from a topological point of view so we want to spend some time developing intuition for these objects. When one starts to sketch continua on a piece of paper, it is hard to imagine a continuum that is not decomposable, and a reader not familiar with these objects might ask whether such continua exist. They do indeed, and are quite common occurrences in chaotic dynamical systems. Fortunately such continua have been studied and their properties explored by mathematicians since 1910.\# All indecomposable continua share certain structure in that

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each can be partitioned into *composants*. If $X$ is an indecomposable continuum and $x \in X$, then $\text{Com}(x)$ is called the *composant* of $x$ and is defined by

$$\text{Com}(x) := \{ y \in X : \text{there is a proper subcontinuum of } X \text{ that contains both } x \text{ and } y \}. $$

The set of composants of an indecomposable continuum partitions the continuum into what can be shown must be an uncountable collection of mutually disjoint connected sets, each of which is dense in the continuum. Each composant is like a “highway” in the continuum. The continuum is made from the collection of highways, each close to any other but forever apart from the other.

Indecomposable continua have been used to describe strange attractors of nonlinear dynamical systems. For example, the Smale horseshoe attractor is indecomposable and can be shown to be homeomorphic to the Knaster bucket handle continuum, one of the more famous indecomposable continua from topology.\(^{10}\) Kuratowski (1968) gives the following constructive description of the Knaster bucket handle, a set of points in the plane we will denote by $\mathbb{K}$. Let $C$ be the Cantor middle-thirds set.\(^{11}\) Note that this set is “symmetric” in the interval, so that if $x \in C$ there is a $y \in C$ such that $(x + y)/2 = 1/2$. This allows one to draw semi-circles in the upper half of the plane (i.e., with non-negative $y$-coordinate) centered at $[1/2, 0]$ that touch the $x$-axis only at points in $C \times \{0\}$. Attached to these semi-circles in the upper half of the plane are semi-circles from below in the lower half plane in the following way: connect “symmetric” points in the intervals $[\frac{2}{3^n}, \frac{1}{3^n}] \times \{0\}$ with semi-circles centered at $[\frac{5}{2} \cdot \frac{1}{3^n}, 0]$. The union of all these semi-circles is the Knaster bucket handle $\mathbb{K}$ (see Figure 4).

This continuum has an uncountable number of composants. To see this, note that $C$ is an uncountable set and $\mathbb{K}$ goes through each point in $C$. Since each composant goes through a countably infinite number points of $C$, there must be an uncountable number of composants. One composant that is easily constructed is the one that goes through all the end points of the Cantor set. This is the only composant that is the one-to-one image of a *ray*, i.e., it has an end-point.\(^{12}\) Starting at the origin, this composant goes up and down through the

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\(^{10}\)One interesting thing to note, is that even if indecomposable continua seem strange, they are “normal” for continua in the sense that most continua are indecomposable. More specifically, Bing (1951) shows that if one considers the set of continua in $Z$ where $Z$ is either $n$-dimensional Euclidean space or the Hilbert cube, then the set of pseudoarcs (a special type of indecomposable continuum) is a dense $G_\delta$ set.

\(^{11}\)The Cantor middle-third set $C$ is a particular example of a Cantor set in the interval $I$. The defining features of a Cantor set are that the set be closed, totally disconnected and perfect in $I$. The set $C$ can be constructed as follows. Let $F_0 = [0, 1]$. To get $F_1$ remove the middle third open segment from each segment of $F_0$, so $F_1 = [0, 1/3] \cup [2/3, 1]$. To get $F_2$, remove the middle third segment from each segment of $F_1$, so $F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this manner to construct a sequence of nested compact sets $F_0 \supset F_1 \supset F_2 \cdots$. Note that each set $F_n$ is the union of $2^n$ sets each with length $(1/3)^n$. The Cantor set $C := \cap_{n=0}^{\infty} F_n$.

\(^{12}\)It is known that each of the other composants is a one-to-one image of the real line. This is a non-trivial result.
x-axis at points \{0, 1, 2/3, 1/3, 2/9, 7/9, 8/9, \ldots \} \times \{0\}. Being an indecomposable continuum, each composant of \( K \) must be dense in the continuum. To think about how they are dense, consider the end-point composant described above in \( K \). This composant goes through all of the end points in \( C \), but this set of end points is dense in \( C \) so it is relatively clear why this composant is dense in the \( K \). Heuristically, each composant is “winding” through the continuum getting arbitrarily close to all other points in the continuum.

Figure 4: Knaster bucket handle. This figure contains part of one of the uncountable number of composants of the Knaster bucket handle. This composant passes through the x-axis at all of the end points of the Cantor set, an countably infinite set. The figure drawn only goes through 32 points in the Cantor set.

Why is the Knaster bucket handle indecomposable? One can show that \( K \) is an arc continuum, i.e., the only proper subcontinua of \( K \) are arcs (connected segments of the composants with finite length). With two such arcs, one cannot cover a single composant alone yet the uncountable number of composants that make up \( K \). Intuitively, a decomposable continuum can be broken into two connected pieces (think of a disk being broken into two connected pieces). If one tries to break \( K \) into two pieces, it will shatter into an uncountable number of disjoint pieces.

Another way of thinking about indecomposable continua in the plane involves an alternative characterization involving the topological concept of nowhere dense and component from continuum theory.

**Definition 6.** A subset \( A \) of a topological space \( X \) is nowhere dense if the closure of \( A \) in \( X \) contains no nonempty open set in \( X \).

**Definition 7.** If \( A \) is a subset of a topological space \( X \), then \( A' \) is a component of \( A \) if \( A \) is connected and \( A' \) is not a proper subset of any connected subset of \( A \).
Consider \(\mathbb{R}^2\) with the topology induced by the Euclidean metric. Now if \(X\) is a continuum contained in the plane, then \(X\) is a topological subspace with open sets of the form \(u \cap X\) where \(u\) is open in \(\mathbb{R}^2\). An alternative way of characterizing indecomposable continua is the following: A continuum \(X\) is indecomposable if and only if every closed, connected proper subset of \(X\) is nowhere dense. What does this mean for indecomposable continua in \(\mathbb{R}^2\)? Intuitively, it says that if \(X\) is an indecomposable continuum in \(\mathbb{R}^2\) and \(u\) is an open set in \(\mathbb{R}^2\) such that \(u \cap X \neq \emptyset\), then \(u \cap X\) has an uncountably infinite number of components.

One way to construct indecomposable continua is through inverse limits. We turn now to our discussion of inverse limits. Again, we want to note that inverse systems and inverse limits can be defined for a much broader class of factor spaces and indexing sets, but the definitions below are sufficient for our purposes.

Let \(I := [0, 1]\) (a nonempty compact metric space) and suppose \(f : I \rightarrow I\) is a continuous function. Let \(Q = [0, 1]^\infty\) be the Hilbert cube and \(\mathbb{Z}^+\) denote the non-negative integers. The space \(I\) is called the factor space and the function \(f\) is called the bonding map. The pair \((X, f)\) is called an inverse system. The set of points

\[
\lim_{\leftarrow}(I, f) := \{x = (x_0, x_1, \ldots) \in Q \mid x_i = f(x_{i+1}) \text{ for } i \in \mathbb{Z}^+\},
\]

is the inverse limit of the inverse system \((I, f)\). Note that \(\lim_{\leftarrow}(I, f)\) is a subset of the Hilbert cube and each point in the inverse limit corresponds to a backward solution to the dynamical system \(f : I \rightarrow I\).\(^{13}\)

The properties of inverse limits have been studied at least since the 1960s and much is known about them. Background theorems we need about their properties are given below. The theorem statements and many of their proofs can be found in Ingram (2000), Nadler (1992), and other books. A nice, well-written introduction to inverse limits on an interval with one bonding map is given in Ingram and Mahavier (2004), along with an investigation of the relationship between the complexity of the topology of the inverse limit space and the complexity of the dynamics on the resulting inverse limit space.

The next theorem tells us that the inverse limit in the CIA model, though a subset of the Hilbert cube, has many nice properties, including the ability to be “embedded” in the plane.

**Theorem 2.** If \((I, f)\) is an inverse system, then the inverse limit of the inverse sequence is a nonempty chainable continuum contained in \(Q\). Furthermore \(\lim_{\leftarrow}(I, f)\) is of topological

\(^{13}\)More generally, one has a sequence of factor spaces \(\{X_1, X_2, \ldots\}\) where \(X_i\) is a nonempty metric space, and a sequence of bonding maps \(\{f_1, f_2, \ldots\}\) such that \(f_i : X_{i+1} \rightarrow X_i\) for \(i \in \mathbb{Z}^+\). In this case, the collection \((X_m, f_m)\) is an inverse system, and the inverse limit is defined in an analogous way. So more carefully, we would say that our inverse system \((I, f)\) should be written as \((X_m, f_m)\) where \(X_m = I\) and \(f_m = f\) for all \(m \in \mathbb{Z}^+\).
dimension 1, can be embedded in the plane, and has the fixed point property.

We need to be able to talk about subsets of $\lim(I, f)$ especially closed subsets of $\lim(I, f)$. If $m \in \mathbb{Z}^+$, the map $\pi_m : \lim(I, f) \to I$ defined by $\pi_m(x) = x_m$ is called the projection map (or, if specificity is required the $m^{th}$ projection map). Each $\pi_m$ is continuous, and the following proposition gives us a way to discuss the subsets of $\lim(I, f)$. Note that this requires us to enlarge the definition of inverse limit space somewhat. Suppose that $A$ is a nonempty subset of $\lim(I, f)$. Let $f_m = f|\pi_{m+1}(A)$ for each $m \in \mathbb{Z}^+$. Define

$$\lim(\pi_i(A), f_i) := \{x = (x_0, x_1, \ldots) \in Q | x_i = f_i(x_{i+1}) \text{ for } i \in \mathbb{Z}^+, x_{i+1} \in \pi_{i+1}(A)\}.$$ **Proposition 7.** Suppose $f : I \to I$ is continuous. If $A$ is a nonempty subset of the inverse limit space $Y := \lim(I, f)$, then $\lim(\pi_n(A), f|\pi_{n+1}(A))$ is a nonempty subset of $Y$ and $A \equiv \lim(\pi_n(A), f|\pi_{n+1}(A))$. Furthermore, if $A$ is a closed, nonempty subset of the inverse limit space $Y := \lim(I, f)$, then $\lim(\pi_n(A), f|\pi_{n+1}(A))$ is a closed, nonempty subset of $Y$ and $A \equiv \lim(\pi_n(A), f|\pi_{n+1}(A))$.

**Proof.** We prove the proposition for a closed subset $A$; the proof is similar for an arbitrary nonempty set $A$. Suppose $A$ is a closed nonempty subset of $Y$. Then $\pi_n(A) \neq \emptyset$ and $\pi_n(A)$ is closed in $X$ for each positive integer $n$. Furthermore, $f_n := f|\pi_{n+1}(A)$ maps $\pi_{n+1}(A)$ onto $\pi_n(A)$. If $f_n$ does not map $\pi_{n+1}(A)$ onto $\pi_n(A)$, there is a point $x_n$ in $\pi_n(A) \setminus f_n(\pi_{n+1}(A))$. But $x_n \in \pi_n(A)$ means that there is some point $y \in A$ such that the $n$th coordinate of $y$ is $x_n$, and $y_{n+1} \in \pi_{n+1}(A)$, and $f(y_{n+1}) = f_n(y_{n+1}) = y_n = x_n$. This is a contradiction, so $f_n$ is onto. If $z$ is a point of $A$, then for each $n$, $z_n \in \pi_n(A)$, so $z \in \lim(\pi_n(A), f|\pi_{n+1}(A))$. Thus, $A \subset \lim(\pi_n(A), f|\pi_{n+1}(A))$ and $\widehat{A} := \lim(\pi_n(A), f|\pi_{n+1}(A))$ is a closed, nonempty subset of $Y$.

For each $n$, let $A_n = \{x \in Y : x_i \in \pi_i(A)\}$. Then each $A_n$ is a closed nonempty subset of $Y$, $A_1 \supset A_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} A_n = \widehat{A}$. If $x \in A_n$, then $d(x, y) \leq 1/2^n$, where $y$ is a point of $A$ whose $n$th coordinate is $x_n$ (which means that $y_i = x_i$ for each $i \leq n$). Then $A = \widehat{A}$.

The next theorem deals with the projection maps $\pi_m$ and how these maps are related to each other via the bonding map $f$.

**Theorem 3.** If $(I, f)$ is an inverse system, and $X := \lim(I, f)$ is the inverse limit, then for each $m < n \in \mathbb{Z}^+$, $\pi_m|X = f^{n-m} \circ (\pi_n|X)$.

The next theorem is our first to discuss the relationship between the bonding map $f$ and the inverse limit $\lim(I, f)$. It provides sufficient conditions for the inverse limit to be extremely simple (topologically).
Theorem 4. If \((I, f)\) is an inverse limit system such that \(f\) is a homeomorphism, then the inverse limit of the inverse system is homeomorphic to an arc. (That is, \(\lim(I, f)\) is homeomorphic to the space \([0, 1]\).)

Sometimes part of an inverse limit can be described as a ray. A topological ray is a locally compact, connected metric space \(R\) containing a point \(O\) such that \(R\setminus\{O\}\) is connected, and if \(p \in R\), but \(p \neq O\), then \(R\setminus\{p\}\) is the union of two disjoint connected sets. Two examples of rays are (1) the set \([0, 1]\) in \(\mathbb{R}\) and (2) the graph of \(\sin(1/x)\) on \((0,1)\) in \(\mathbb{R}^2\). The next theorem gives sufficient conditions for a union of arcs to be a ray.

Theorem 5. If \(\alpha_1, \alpha_2, \ldots\) is a sequence of arcs each of which is a proper subset of a continuum \(X\) such that \(\alpha_1 \subset \alpha_2 \subset \alpha_3 \subset \cdots\), the point \(O\) is a common endpoint of \(\alpha_1, \alpha_2, \ldots\), \(R = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \cdots\), and no point of \(\alpha_n\) belongs to \(\overline{R\setminus\alpha_{n+1}}\), then \(R\) is a ray.

Theorem 6. (Subcontinua) Suppose \((I, f)\) is an inverse limit sequence. If, for each \(m\), \(K_m\) is a subcontinuum of \(I\) and \(f(K_{m+1}) = K_m\), then \(\lim(K_m, f|K_{m+1})\) is a subcontinuum of \(\lim(I, f)\).

Let \(X = \lim(I, f)\). In this case, a natural map is induced on the inverse limit space by the bonding map \(f\): for \(x = (x_0, x_1, \ldots) \in X\), define \(F(x) = F((x_0, x_1, \ldots)) = (f(x_0), f(x_1), \ldots) = (f(x_0), x_0, x_1, \ldots)\). The induced map \(F\) is a homeomorphism from \(X\) onto \(X\). The inverse \(\sigma := F^{-1}\) of \(F\), is then defined by \(\sigma(x) = \sigma((x_0, x_1, \ldots)) = (x_1, x_2, \ldots)\). Thus, the pair \((X, F)\) forms a dynamical system, one that runs both forward and backward. The induced map \(\sigma\) is called the shift homeomorphism. Note that we have, in a sense, “turned” a continuous map \(f\) on a space \(I\) into a homeomorphism \(F\) on a possibly more complicated space \(X\).

The theorems below from Ingram (1995, 2002, 2003) deal with properties of the bonding map \(f\) and the associated inverse limit or periodic points of \(f\). The properties of the bonding map include monotone, unimodal, and type 1. We define these properties next.

Definition 8. A map of a continuum to itself is monotone provided each point inverse is a continuum. A map \(f\) of an interval \([a, b]\) onto itself is unimodal provided \(f\) is not monotone, and there is a point \(c\) in \((a, b)\) such that \(f(c) \in \{a, b\}\) and \(f|[a, c]\) and \(f|[c, b]\) are both monotone. The map \(f\) is a type (1) unimodal map if \(f(b) = a\).

Note that in case I.A, the backward map from the CIA model is unimodal. Now on to the theorems.

Theorem 7 (Ingram (1995)). Suppose \(f\) is a type (1) unimodal mapping of an interval \([a, b]\) onto itself with critical point \(c\), and \(q\) is a point in \((c, p]\) such that \(f^2(q) = q\) and \(f(a) = q\). Then the inverse limit of the inverse limit system \(([a, b], f)\) is the union of two Knaster bucket handle continua intersecting at a point or an arc.
Theorem 8 (Ingram (1995)). Suppose $f$ is a type (1) unimodal mapping of an interval $[a, b]$ onto itself and $q$ is the first fixed point for $f^2$ in $[c, b]$. Then $f$ has a periodic point of odd period greater than 1 if and only if $f^2(b) < q$.

Theorem 9 (Ingram (1995)). Suppose $f$ is a type (1) unimodal mapping of an interval $[a, b]$ onto itself and $q$ is the first fixed point for $f^2$ in $[c, b]$. Then $\lim (\{a, b\}, f)$ is indecomposable if and only if $f(a) < q$.

Theorem 10 (Ingram (2002)). Suppose $f : [a, b] \rightarrow [a, b]$ is a continuous mapping, $a$ is periodic of period $n \geq 3$ under $f$ and $b$ is in $O_+(a)$. If $k$ is an integer such that $f^k(a)$ is the first member of $O_+(a) \setminus \{a\}$, and $n$ and $k$ are relatively prime, then $\lim (\{a, b\}, f)$ is an indecomposable continuum. (Note: $f^k(a)$ is the first member of $O_+(a) \setminus \{a\}$ means first relative to the order on the interval $[a, b]$.)

Theorem 11 (Ingram (2003)). Suppose $f : [a, b] \rightarrow [a, b]$ is a continuous mapping and is a Markov map with Markov partition $a = a_1 < a_2 < \cdots < a_n = b$ for $n \geq 3$ and $O_+(a) = \{a_1, a_2, \ldots, a_n\}$. If $k$ is an integer, $k < n$, such that $f^k(a) = a_2$, and $n$ and $k$ are relatively prime, then $\lim (\{a, b\}, f)$ is an arc continuum.

The most carefully studied families of bonding maps are the so-called tent maps and logistic maps. [See Barge et al. (1996), Barge and Diamond (1998), Ingram (1995), and Ingram and Mahavier (2004), for example.] Different authors define them slightly differently, but loosely, all the families below except for the last are tent maps. The $h_\lambda$, $f_\lambda$, $P_\lambda$, and $Q_\lambda$ families are type (1) unimodal maps on the unit interval. Define families of maps from $[0, 1]$ to $[0, 1]$ as follows:

1. Suppose $\lambda \in [1, 2]$. For $x \in [0, 1]$,
   \[ T_\lambda(x) = \begin{cases} \lambda x, & 0 \leq x \leq 1/2 \\ \lambda(1 - x), & 1/2 \leq x \leq 1 \end{cases}. \]

2. Suppose $\lambda \in [1, 2]$. For $x \in [0, 1]$,
   \[ h_\lambda(x) = \begin{cases} \lambda x, & 0 \leq x \leq 1/\lambda \\ 2 - \lambda x, & 1/\lambda \leq x \leq 1 \end{cases}. \]

3. Suppose $\lambda \in [1, 2]$. For $x \in [0, 1]$,
   \[ f_\lambda(x) = \begin{cases} \lambda x + (2 - \lambda), & 0 \leq x \leq \frac{\lambda - 1}{\lambda} \\ -\lambda x + \lambda, & \frac{\lambda - 1}{\lambda} \leq x \leq 1 \end{cases}. \]
4. Suppose $\lambda \in [0, 1]$. For $x \in [0, 1]$, 

$$P_{\lambda}(x) = \begin{cases} 
2x, & 0 \leq x \leq \frac{1}{2} \\
2(\lambda - 1)x + \lambda, & \frac{1}{2} \leq x \leq 1
\end{cases}.$$  

5. Suppose $\lambda \in [0, 1]$. For $x \in [0, 1]$, 

$$Q_{\lambda}(x) = \begin{cases} 
2(1 - \lambda)x + \lambda, & 0 \leq x \leq \frac{1}{2} \\
2(1 - x), & \frac{1}{2} \leq x \leq 1
\end{cases}.$$  

6. Suppose $\lambda \in [0, 1]$. For $x \in [0, 1]$, 

$$L_{\lambda}(x) = 4\lambda x(1 - x).$$

Type (1) unimodal maps go up and then come down; members of our family of maps go down and then come up. This is not a problem: we can “flip” our map over so as to more easily use the results in the literature. Also, translating to $[0, 1]$ from the interval $[a, b] \subset (0, \infty)$ is easy. This is made explicit in the proposition below.

**Proposition 8.** Suppose $f : [a, b] \rightarrow [a, b]$ has the following properties:

(a) There is some $c$ in $(a, b)$ such that $f|[a, c]$ is strictly decreasing, and $f$ maps $[a, c]$ onto $[a, b]$.

(b) The map $f|[c, b]$ is strictly increasing.

1. Let $d = f(b)$. Then $h : [a, b] \rightarrow [0, 1]$ defined by $h(x) = \frac{x - a}{b - a}$ is a homeomorphism and $g = h \circ f \circ h^{-1}$ is a map from $[0, 1]$ onto $[0, 1]$. Note that $h(a) = 0$ and $h(b) = 1$. Let $c_g = h(c) = \frac{e - a}{b - a}$. The map $g$ is strictly decreasing on $[0, c_g]$ and strictly increasing on $[c_g, 1]$. If $f|[c, b]$ is linear, then $g|[c_g, 1]$ is linear. Thus, $f$ is conjugate to the map $g : [0, 1] \rightarrow [0, 1]$.

2. If $H(x) = 1 - x$ for $x \in [0, 1]$, $H : [0, 1] \rightarrow [0, 1]$ is a homeomorphism such that $H^{-1} = H$. Let $k = H \circ g \circ H$. Note that $H(0) = 1$, $H(1) = 0$, and $H(c_g) = 1 - c_g$. The map $k|[1 - c_g, 1]$ is strictly decreasing and $k|[0, 1 - c_g]$ is strictly increasing. If $f$ is linear on $[c, b]$, then $k$ is linear on $[0, 1 - c_g]$.

Even with the translation and flip, members of our family of maps differ in one respect from those of the tent family above. We have contraction on one part of the interval, but on the remainder of the interval, we may have expansion or a mixture of expansion and contraction. It is easy to see that no member of the family we are investigating can be conjugate to a member of the $T_\lambda$, $h_\lambda$, and $P_\lambda$ families. Whether a member of our family...
can be conjugate to a member of the $f_\lambda$ family is not clear. A member of our family can be conjugate to a member of the $Q_\lambda$ family; that is proven in the section 5.

The concept of conjugacy mentioned earlier in the discussion on dynamics is also an extremely useful property for understanding inverse limits. For if two maps $f : X \to X$ and $g : Y \to Y$ are conjugate, then their inverse limits $\lim(X, f)$ and $\lim(Y, g)$ are homeomorphic as well. More specifically, if one knows that $\lim(Y, g)$ is indecomposable and the backward map from the CIA model $f$ is conjugate to $g$ then the inverse limit $\lim(X, f)$ from the CIA model is also indecomposable.

4 Why Inverse Limits?

As mentioned in the introduction, the earlier methods of dealing with ill-defined forward dynamics are unsatisfactory for at least two reasons: (1) the orbits from the (well-defined) backward map are going in the wrong direction for discussing equilibria, and (2) interesting possible dynamics are ignored when a local analysis around a steady state is used. Since the dynamics in the CIA model are not well-defined going forward in time concepts like orbits and chaotic need to be modified for such a dynamical system. We will see that the theory of inverse limits is the right tool for understanding the forward dynamics even though they are not well-defined in the model.

Let $J = [x, \bar{x}]$ and consider case I.A. Using the implicit difference equation $A(x_{t+1}) = B(x_t)$ one can define the backward map $f : J \to J$ and the forward correspondence $f^{-1} : J \to J$ (technically this is a map from sets of $J$ into the power set of $J$). The orbit of $x \in J$ under the action of $f$ is defined in the usual way $O^f(x) = \{x, f(x), f^2(x), \ldots\}$. Since the dynamics are not well-defined going forward in time, there is not an orbit in the usual sense of the term (likewise, there is no backward orbit of $x$ under the action of $f$). One can think of perhaps many forward orbits of $x$ under the action of $f^{-1}$ (or perhaps many backward orbits of $x$ under the action of $f$). We can identify these orbits with points in the inverse limit space $X := \lim(J, f)$:

$O^f_{-1}(x) := \{x \in X | \pi_0(x) = x\}$.

Let $F : X \to X$ be the homeomorphism induced by $f$ and let $\sigma := F^{-1}$ be its inverse. An equilibrium in the CIA model corresponds precisely to points in the inverse limit $x \in X$, i.e. a sequence of numbers

$x = \{(x_0, x_1, x_2, \ldots) | x_{n+1} \in f^{-1}(x_n), n = 0, 1, \ldots\}$.

Next, we see that equilibria in the CIA will exhibit certain properties iff the homeomorphism $\sigma : X \to X$ has these properties. There is a periodic equilibrium of period $N$ in the
CIA model \( x = (x_0, x_1, \ldots, x_N, x_0, x_1, \ldots) \) iff \( x \) is a periodic point under \( \sigma \), i.e.,

\[ O^*_+(x) = \{ x, \sigma(x), \ldots, \sigma^{N-1}(x), x, \sigma(x), \ldots \}. \]

There is an equilibrium in the CIA model \( x = (x_0, x_1, x_2, \ldots) \) where \( \{x_0, x_1, x_2, \ldots\} \) is dense in \( J \) iff the orbit of \( x \) under the action of \( \sigma \) is dense in \( X \), i.e., \( O^*_+(x) = \{ x, \sigma(x), \sigma^2(x), \ldots \} \) is dense in \( X \). Furthermore, the set of periodic points in the periodic equilibria of the CIA is dense in \( J \) iff the set of periodic points in \( X \) under \( \sigma \) is dense in \( X \).

We can also talk about sensitive dependence on initial conditions in the CIA model. We say that equilibria in the CIA model have sensitive dependence on initial conditions if there exists \( \delta > 0 \) such that for a given equilibrium \( x \) and any neighborhood \( N \) of \( x_0 \), there exists another equilibrium \( y \in X \) with \( y_0 \in N \) and \( n \geq 0 \) such that \( d(x_n, y_n) > \delta \). In terms of the inverse limit space, we say that \( \sigma : X \rightarrow X \) exhibits sensitive dependence on initial conditions if there exists \( \delta > 0 \) such that for any \( x \in X \) and \( \epsilon > 0 \) there exists a \( y \in X \) and \( n \geq 0 \) such that \( d(x, y) < \epsilon \) and \( d(\sigma^n(x), \sigma^n(y)) > \delta \). Note that the equilibria in the CIA have sensitive dependence on initial conditions iff the map \( \sigma : X \rightarrow X \) has sensitive dependence on initial conditions.

From this discussion, we see that understanding the forward dynamics in the CIA model (even though the map is not well-defined) is equivalent to understanding the dynamics of the shift map which is well-defined on the inverse limit space \( \lim \leftarrow (J, f) \). It is for this reason that we believe inverse limits are the right tool for thinking about equilibria in models with ill-defined forward dynamics.

The definition of chaotic given by Devaney, can be adapted to the CIA model where the forward map is not well-defined in the following manner. We say the equilibria in the CIA model are chaotic in the sense of Devaney (2003) if

A1: there is an equilibrium in the CIA model \( x = (x_0, x_1, x_2, \ldots) \) where \( \{x_0, x_1, x_2, \ldots\} \) is dense in \( J \);

A2: the set of points in the periodic equilibria is dense in \( J \);

A3: the equilibria in the CIA model have sensitive dependence on initial conditions if there exists \( \delta > 0 \) such that for a given equilibrium \( x \) and any neighborhood \( N \) of \( x_0 \), there exists another equilibrium \( y \in X^* \) with \( y_0 \in N \) and \( n \geq 0 \) such that \( d(x_n, y_n) > \delta \).

The map \( \sigma : X \rightarrow X \) is chaotic in the sense of Devaney (2003) if

B1: there is a point \( p \in X \) such that its orbit under the action of \( \sigma \) is dense in \( X \);

B2: the set of periodic points in \( X \) is dense in \( X \);
B3: the map $\sigma : X \to X$ has sensitive dependence on initial conditions on the invariant closed subset $H$ of $X$ if for each there is some positive number $r$ such that for each point $x$ in $X$ and for each $\epsilon > 0$, there is a point $y$ in $X$ with $d(x, y) < \epsilon$ and an integer $k \geq 0$ such that $d(\sigma^k(x), \sigma^k(y)) \geq r$.

Since $A1$ iff $B1$, $A2$ iff $B2$, and $A3$ iff $B3$, we see that equilibria in the CIA model are chaotic in the sense of Devaney iff the the shift map on the inverse limit space $\sigma : X \to X$ where $X := \lim(I, f)$ is chaotic in the sense of Devaney. Now, the question is how do we determine if $\sigma$ on the inverse limit space is chaotic? In particular, how are $f$, $F$ and $\sigma$ related?

Suppose that $X$ and $Y$ are metric spaces, $f : X \to X$ is continuous and $g : Y \to Y$ is continuous. If there is a continuous map $h : X \to Y$ such that $h \circ f = g \circ h$, then $f$ and $g$ are said to be semiconjugate, and $f$ factors over $g$. If there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, then $f$ and $g$ are said to be conjugate. Whenever two maps are conjugate, their dynamics are equivalent. If $f$ and $g$ are semiconjugate then their respective dynamics are related, but not necessarily equivalent. What then is the relationship between $f : I \to I$, and the induced homeomorphism $F : \lim(I, f) \to \lim(I, f)$? For each nonnegative integer $m$,

$$\lim(I, f) \xrightarrow{F} \lim(I, f)$$

$$\downarrow \pi_m \quad \downarrow \pi_m$$

$$I \xrightarrow{f} I$$

is a semiconjugacy, i.e., $\pi_m \circ F = f \circ \pi_m$. Thus, although $F$ on $\lim(I, f)$ is only semiconjugate to $f$ on $I$, it is a very strong semiconjugacy - almost an “$\epsilon$-semiconjugacy”: For each $\epsilon > 0$, there is a nonnegative integer $M$ such that if $m > M$, $Y_m = \{x = (x_0, x_1, \ldots, x_m) : f(x_{i+1}) = x_i \text{ for } x_i \in I, i = 0, \ldots, m - 1\}$ is homeomorphic to $I$, $Y_m$ is homeomorphic to $Y^*_m := \{x = (x_0, x_1, \ldots, x_m, 0, 0, 0, \ldots) : f(x_{i+1}) = x_i \text{ for } i = 0, \ldots, m - 1\} \subset Q$, and the Hausdorff distance from $Y^*_m$ to $\lim(I, f)$ is less than $\epsilon$. Furthermore, if we define $\gamma : Y_m \to Y^*_m$ by

$$\gamma(x) = \gamma((x_0, x_1, \ldots, x_m)) = (f(x_0), f(x_1), \ldots, f(x_{m-1})),$$

$\delta : Y^*_m \to Y^*_m$ by

$$\delta(x) = \delta((x_0, x_1, \ldots, x_m, 0, 0, 0, \ldots) = (f(x_0), f(x_1), \ldots, f(x_{m-1}), 0, 0, \ldots),$$

and $h_m : \lim(I, f) \to Y^*_m$ by

$$h_m(x) = h_m(x_0, x_1, \ldots) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots).$$
Then $\gamma$ is conjugate to $f$, $\gamma$ is conjugate to $\delta$:

$$\lim(I, f) \xrightarrow{F} \lim(F, I)$$

$$\begin{array}{cccc}
h_m & \downarrow & \downarrow & h_m \\
Y^*_m & \delta & \pi[0, m] & Y^*_m \\
p[0, m] & \downarrow & \downarrow & \pi[0, m] \\
Y_m & \gamma & \pi_m & Y_m \\
p_m & \downarrow & \downarrow & \pi_m \\
I & f & I &
\end{array}$$

We use $\pi_{[0, m]}$ to denote the projection map $\pi_{[0, m]}(x) = (x_0, x_1, \ldots, x_m)$. Note that $h_m \circ F = \delta \circ h_m$ with $\text{diam}(h_m^{-1}(x)) < 1/2^m \forall x \in Y^*_m$. The homeomorphism $F$ is “$\epsilon$-semiconjugate” to $f$ in the sense that

1. $\forall m, \pi_m \circ \pi_{[0, m]} \circ h_m \circ F = f \circ \pi_m \circ \pi_{[0, m]} \circ h_m$ and

2. if $x \in I$, then $h_m^{-1}(\pi_{[0, m]}^{-1}(\pi^{-1}_m(x)))$ has diameter less than $\epsilon$.

Thus, given that $F$ is a homeomorphism and $f$ is not (generally) a homeomorphism, and therefore $F$ cannot (generally) be conjugate to $f$, $F$ and $f$ are related by semiconjugacies that are “close” to being conjugacies. Furthermore, if $F$ and $f$ are closely related, then we can expect $\sigma$ to be closely related to $f^{-1}$, which is not even a well-defined map from $I$ to $I$ in general, but which does represent the forward-in-time dynamics that interest us.

### 5 An Application of Inverse Limits

To establish the existence of chaotic equilibria in the CIA model, [MR] use the Li-Yorke theorem that asserts a period three orbit implies chaos (in the sense of Li and Yorke). However, this is not the end of the story. In this section, we apply the results from section 3 to the cash-in-advance model described in section 2. Depending on the case, we focus on the interval $J$, since any solutions that contain a point outside this interval behave in a simple way. Note that $f|J : J \to J$ is surjective. We denote $f|J$ as just $f$, since it should not lead to confusion.

If we consider the map $f : J \to J$ and form the inverse limit $X = \lim(J, f)$, $X$ is a chainable continuum and can therefore be realized as a subset of the plane. The points of $X$ are precisely the forward solutions of the implicitly-defined difference equation $A(x_{t+1}) = \ldots$
$B(x_t)$ that stay in $J$. Let $F : X \to X$ be the induced homeomorphism by $f$. Since $F$ is a homeomorphism from $X$ to itself, $\sigma := F^{-1}$ is also a homeomorphism from $X$ to itself. Specifically, $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. Thus, $\sigma$ is actually a shift map. Note that applying $F$ corresponds to going backward in time, while applying $\sigma$ corresponds to going forward in time. Thus, by going to a different space, one formed naturally by the difference equation solutions, we have essentially turned $f$ into a homeomorphism (at the expense of dealing with a perhaps more complicated space than just the interval).

How complicated topologically the inverse limit space is, is a measure of the complexity of the dynamics of the original $f$ on $J$. We prove below that the inverse limit space $X$ can be as simple as an arc, so that the induced homeomorphism $F$ is just a homeomorphism on an arc. It can be a double-sided topologist’s sine curve limiting on two arcs. [See Ingram and Mahavier (2004) for a picture of the double topologist’s $\sin(1/x)$ curve.] Even for the period three point case, the space $X$ may have a dense set of periodic points; alternatively, it may have a dense set of points contained in the basin of attraction of a period three orbit. In the latter case, the chaotic behavior is completely contained in an invariant Cantor set in $J$ (an uncountable set of measure zero). The space $X$ can be the union of two indecomposable continua intersecting in a point or an arc. Many things can happen; this list is not exhaustive.

It can most likely (see discussion in the last subsection) be so complicated that $X$ itself is not only indecomposable, but $X$ also contains a copy of every member of the family of inverse limit spaces obtained from tent families on the interval. [See Barge et al. (1996).] In this case it would contain an uncountable number of topologically different indecomposable subcontinua, and it is very far from being an arc continuum.

The next proposition deals with cases I.B, I.C, II.B, and II.C and shows that the dynamics are simple in these cases.

**Proposition 9.** Consider cases I.B, I.C, II.B, and II.C. Let $(J, f)$ be the inverse system. The inverse limit is an arc or a point.

**Proof.** In these cases, $f$ is a homeomorphism so by Theorem 4, the inverse limit is homeomorphic to $J$. 

In the next two subsections, we deal with case I.A. In particular, we show that the dynamics may be complicated on a topologically large set, complicated on a topologically small set, or rather simple.
5.1 Period Three Point with(out) a Dense Set of Periodic Points

Applying the results discussed in the previous section, we see that if \( \{x, x, c\} \) forms a period three orbit for \( f \), \( X \) is an indecomposable continuum. Furthermore, it follows from Sarkovskii’s Theorem that \( f \) has periodic orbits of all periods; and therefore, \( \sigma : X \to X \) admits periodic orbits of all periods. However, \( f \) may be chaotic only on an invariant Cantor set contained in the interval, or it may have a dense set of periodic points (and thus be chaotic on the entire interval), or perhaps it could be something in between. We show that the first two cases can occur. See Figure 5 for a map \( g : [0,1] \to [0,1] \) with a period 3 orbit conjugate (via a translation and flip) to a backward map from the CIA model.

Figure 5: A period 3 orbit for a map \( g \) on \([0,1]\) conjugate to a backward map from the CIA model.

It can easily be the case that \( f \) is contracting on \([c, x]\). If \( f \) is expanding on \([x, c]\), and that expansion is large enough to dominate the contraction on the rest of the interval, then \( f^2 \) is expanding on the entire interval, and it follows that the set of periodic points in the interval is dense:

Suppose \( J \) denotes a subinterval of \( \mathbb{R} \). A function \( f : J \to \mathbb{R} \) is piecewise monotone if and only if \( f \) is continuous and there are a finite number of subintervals \( J_1, J_2, \ldots, J_n \) covering \( J \) such that \( f \) is either strictly increasing or strictly decreasing on each \( J_i \). Note that a piecewise monotone function cannot be constant on any nontrivial interval, and that our unimodal maps on the interval (in the nontrivial cases) are piecewise monotone. We need some results from a paper by Baldwin (1990). Baldwin’s paper deals with piecewise monotone maps on the interval, and his definitions and results are phrased in those terms. Baldwin’s theorem stated below has been re-phrased for our simpler case; Baldwin’s actual theorem is much more general. His definition of itinerary has also been slightly re-phrased, but our definition is equivalent to his. A piecewise monotone function on an interval \( I = [a, b] \)
is expanding if whenever \( x < y \) and \( f \) is monotone on \([x, y]\), then
\[
\left| \frac{f(y) - f(x)}{y - x} \right| > 1.
\]
If \( J = [r_n, r_1] \cup [r_1, r_2] \cup \cdots \cup [r_{n-1}, r_n] \), and \( f : J \to J \) is strictly increasing or strictly decreasing on each subinterval \([r_{i-1}, r_i]\), then let \( T = \{J_0, J_1, \ldots, J_{2n}\} \), where \( J_{2i} = \{r_i\} \) for \( 0 \leq i \leq n \), and \( J_{2i+1} = (r_i, r_{i+1}) \) for \( 0 \leq i \leq n - 1 \). We say \( x \) and \( y \) have different itineraries if \( f^n(x) \) and \( f^n(y) \) are in different members of \( T \) for some \( n \geq 0 \). A piecewise monotone map on an interval \( J \) is weakly expanding if, whenever \( x \neq y \) in \( J \), \( x \) and \( y \) have different itineraries.

**Proposition 10 (Baldwin (1990, Proposition 10)).** An expanding piecewise monotone function on an interval \( I \) is weakly expanding.

**Theorem 12 (Baldwin (1990, Theorem 11)).** Suppose that \( I = [a, b] \), \( a < \gamma < b \), \( f : I \to I \) is surjective and continuous, \( g : I \to I \) is surjective and continuous, both \( f \) and \( g \) are strictly decreasing on \([a, \gamma]\), and both \( f \) and \( g \) are strictly increasing on \([\gamma, b]\). If \( f \) and \( g \) are weakly expanding then \( f \) is conjugate to \( g \). (The statement of this result has been modified to suit our purposes here. Baldwin’s theorem is more general. The function \( f \) and \( g \) above satisfy Baldwin’s \( E_f = E_g \) condition.)

The proposition below is known. We need it for Lemma 3, so its proof is included here for completeness.

**Proposition 11.** Define \( G : [0, 1] \to [0, 1] \) as follows: \( G(x) = 1 - 2x \) for \( x \in [0, \frac{1}{2}] \) and \( G(x) = x - \frac{1}{2} \) for \( x \in [\frac{1}{2}, 1] \). (Thus, \( \{0, \frac{1}{2}, 1\} \) is a period 3 orbit for \( G \).) Then the set of all points in \([0, 1]\) that have a dense orbit is a residual set in \([0, 1]\).

**Proof.** Suppose \( u \) is an open interval in the space \([0, 1]\). (Note that \( u \) may be of the form \([0, c]\) or \((c, 1]\), as well as of the form \((a, b]\).) We want to show that for some \( n \), \( G^n(u) = [0, 1] \). Suppose this is not the case, i.e., that there is a nonempty open interval \( u \) in \([0, 1]\) such that \( G^n(u) \subseteq \mathbb{Q}, 0, 1 \) for each \( n \). Then \( G^n(u) \) does not contain \([0, \frac{1}{2}] \) for any \( n \), for this would mean that \( G^{n+1}(u) = [0, 1] \). Also, \( G^n(u) \) does not contain \([\frac{1}{2}, 1] \) for any \( n \), for this would mean that \( G^{n+2}(u) = [0, 1] \).

Now \( G^n(u) \) must contain a point in \( \{0, \frac{1}{2}, 1\} \) for some \( n \), for if it does not, \( G^n(u) \subseteq (0, \frac{1}{2}) \) or \( G^n(u) \subseteq (\frac{1}{2}, 1) \) for each \( n \). But whenever \( G^m(u) \subseteq (0, \frac{1}{2}) \), the length of \( G^{m+1}(u) \) is twice the length of \( G^m(u) \); whenever \( G^m(u) \subseteq (\frac{1}{2}, 1) \), \( G^{m+1}(u) \subseteq (0, \frac{1}{2}) \), the length of \( G^{m+1}(u) \) is the same as the length of \( G^m(u) \), and the length of \( G^{m+1}(u) \) is twice the length of \( G^m(u) \). Thus, the length of \( G^m(u) \) keeps doubling as \( m \to \infty \). This is a contradiction, and for some \( n \), \( G^n(u) \) contains a point of \( \{0, \frac{1}{2}, 1\} \). Then \( G^n(u) \cup G^{n+1}(u) \cup G^{n+2}(u) \) contains \( \{0, \frac{1}{2}, 1\} \).
Without loss of generality, assume $G^n(u)$ contains 0. Then $G^n(u)$ contains $[0, d)$ for some $d < \frac{1}{2}$. Again, we have a contradiction: The length of $G^{m+n}([0, d))$ keeps doubling as $m \to \infty$. Then for some $k$, $G^k(u)$ contains $[0, \frac{1}{3}]$ or $[\frac{1}{2}, 1]$, and $G^{k+2}(u)$ contains $[0, 1]$. Then $G$ is transitive on $[0, 1]$ and the result follows.

Next we construct period 3 orbit maps on an interval that satisfy the model conditions. One has a dense set of points with dense trajectories. The other has an invariant Cantor set with simple dynamics for the points not in the Cantor set.

**Lemma 1.** Suppose that $K = [a, b]$ is an interval, $h : K \to K$ is continuous, such that $a < c < b$, and $h(a) = b$, $h(b) = c$, and $h(c) = a$ (so that $\{a, c, b\}$ forms a period three orbit in $K$). Suppose $h|([a, a_1) \cup (a_1, c) \cup (c, b)]$ is continuously differentiable, strictly increasing on $[c, b]$, strictly decreasing on $[a, c]$ with $f'(a_1) < f'(a_1^-)$, where $a_1 \in (a, c)$ is the unique solution to $h(a_1) = c$. Suppose further there is $\delta > 0$ such that

1. $0 < h'(y) < 1 - \delta$ for $y \in (c, b]$,
2. $h'(x)h'(y) < -1 - \delta$ for $x \in [a, c) \setminus \{a_1\}$ and $y \in (c, b]$,
3. $h'(a_1^-)h'(y) < -1 - \delta$ and $h'(a_1^-)h'(y) < -1 - \delta$ for $y \in (c, b]$.

Then $h^2 = h \circ h$ is expanding on $[a, b]$.

**Proof.** To simplify notation, when considering inequalities we will let $h'(a_1)$ equal either $h'(a_1^+)$ or $h'(a_1^-)$ and require the inequality to hold for both values. Consequently we will consider $h'(-)$ on $[a, c)$ acknowledging that $h'(a_1)$ takes on two values.

1. For $x \in (a, c)$, $x = h(w)$ for some $w \in (c, b)$. Then $h'(x)h'(w) < -1 - \delta$ and $0 < h'(w) < 1 - \delta$ by assumption. Therefore

$$h'(x) = \frac{h'(x)h'(w)}{h'(w)} < \frac{-1 - \delta}{h'(w)}.$$ 

Since $\frac{1}{h'(w)} > 1$, we have $\frac{1 + \delta}{h'(w)} < 1 + \delta$, which gives $-(1 + \delta) > -\frac{(1 + \delta)}{h'(w)}$. Therefore $h'(x) < -1 - \delta$.

2. Recall $a_1$ denotes that point in $(a, c)$ such that $h(a_1) = c$. Then $h^2$ is expanding on $[a, a_1]$: For $x \in (a, a_1)$, $Dh^2(x) = h'(h(x))h'(x)$ (chain rule) and for $x \in (a, a_1)$, $h(x) \in (c, b)$. Therefore $h'(h(x))h'(x) < -1 - \delta$, and it follows from the Mean Value Theorem that $h^2$ is expanding on $[a, a_1]$.

3. $h^2$ is expanding on $[a_1, c]$: For $x \in (a_1, c)$, $Dh^2(x) = h'(h(x))h'(x)$ with $x \in (a_1, c)$, $h(x) \in (a, c)$. Therefore $h'(h(x)) < -1 - \delta$, $h'(x) < -1 - \delta$, and $h'(h(x))h'(x) > (1 + \delta)^2 > 1$. It follows from the Mean Value Theorem that $h^2$ is expanding on $[a_1, c]$.
Figure 6: Backward map $h$ satisfying the hypotheses of Theorem 1 and Lemma 1. The dynamics in this case contain a dense set of of points with dense orbits.

(4) $h^2$ is expanding on $[a, c]$: Suppose $x \in [a, a_1), y \in (a_1, c]$ (the other cases have been taken care of). Then $x < a_1 < y$ and $h(x) > h(a_1) = c > h(y)$. Since $h^2$ is expanding on $[a, a_1]$, 
\[
\frac{|h^2(a_1) - h^2(x)|}{|a_1 - x|} > 1, \quad \text{and} \quad h^2(x) - h^2(a_1) > a_1 - x. \]  
Likewise, $h^2(a_1) - h^2(y) > y - a_1$. Then 
\[
(h^2(x) - h^2(a_1)) + (h^2(a_1) - h^2(y)) > (y - a_1) + (a_1 - x), \]  
so that $h^2(x) - h^2(y) > y - x$, and 
\[
\frac{h^2(x) - h^2(y)}{y - x} = \frac{|h^2(y) - h^2(x)|}{|y - x|} > 1. \]  
It follows that $h^2$ is expanding on $[a, c]$.

(5) $h^2$ is expanding on $[c, b]$: For $x \in (c, b)$, $h(x) \in (a, c)$ and $Dh^2(x) = h'(h(x))h'(x) < -1 - \delta$. Then the Mean Value Theorem implies $h^2$ is expanding on $[c, b]$.

Thus, $h^2$ is expanding.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Backward map $h$ satisfying the hypotheses of Theorem 1 and Lemma 1. The dynamics in this case contain a dense set of of points with dense orbits.}
\end{figure}

\begin{exm}
In this example, we show that a backward map from the CIA model may satisfy the hypotheses of Lemma 1. Let $a = 1$, $b = 2$, $\bar{c} = 4/3$, and $c_1 = (a + c)/2$. The backward map $h : [a, b] \to [a, b]$ is given by:

\[
h(x) := \begin{cases} 
  b + m_1(x - a) & a \leq x \leq \bar{c}_1 \\
  c + m_2(x - c_1) & \bar{c}_1 \leq x \leq \bar{c} \\
  \gamma + (1-\delta)x - \alpha x^2 & \bar{c} < x < b 
\end{cases}
\]

Let $\delta = 0.1$, $m_1 = -4$, $m_2 = -2$, $\alpha = 3/25$ and $\gamma = 1/75$. This backward map satisfies the hypotheses in Theorem 1. By construction $\{a, \bar{c}, b\}$ is a three cycle. The function $h$ plotted in Figure 6. Note that for $x \in (\bar{c}, b)$, $h'(x) = (1 - \delta) - 2\alpha x$ so we have $29/50 = (1 - \delta) - 2\alpha \bar{c} < h'(x) < 1 - \delta = 0.9$. Furthermore, for $x \in [a, \bar{c}) \setminus \{c_1\}$ we have $h'(x) \leq -2$.

This implies 
\[
h'(x)h'(y) \leq (-2)(29/50) = -29/25 = -1.16 < -1.1 = -1 - \delta.
\]

for $x \in (a, \bar{c})$ and $y \in (\bar{c}, b)$.

\end{exm}
Lemma 2. Suppose that $K = [a, b]$ is an interval, $h : K \to K$, $a < c < b$, and $h(a) = b$, $h(b) = c$, and $h(c) = a$ (so that $\{a, c, b\}$ forms a period three orbit in $K$). Furthermore, suppose $h$ is strictly decreasing on $[a, c]$ and strictly increasing on $[c, b]$. If $h^2$ is expanding on $[a, b]$, then $h$ is weakly expanding on $[a, b]$.

Proof. Let $T = \{(a), (a, c), \{c\}, (c, b), \{b\}\}$, and let $T_0 = \{a\}$, $T_1 = (a, c)$, $T_2 = \{c\}$, $T_3 = (c, b)$, $T_4 = \{b\}$ (so $T = \{T_0, T_1, T_2, T_3, T_4\}$). Then $h$ is monotone on each set in $T$. Suppose $a_1$ is that point in $(a, c)$ such that $h(a_1) = c$. Let

$$S = \{(a), (a, a_1), \{a_1\}, (a_1, c), \{c\}, (c, b), \{b\}\},$$

where $S_0 = \{a\}$, $S_1 = (a, a_1)$, $S_2 = \{a_1\}$, $S_3 = (a_1, c)$, $S_4 = \{c\}$, $S_5 = (c, b)$, and $S_6 = \{b\}$. Then $h^2$ is monotone on each set in $S$. For $x \in [a, b]$, the itinerary of $x$ under $h^2$ is

$$a(x) = (a_0(x), a_1(x), \ldots),$$

where $a_i(x) \in S$ for $i \geq 0$, $a_i(x) = S_n$ means $h^2(x) \in S_n$. Since $h^2$ is expanding on $K$, $h^2$ is weakly expanding on $K$. Therefore, $x \neq y$ in $[a, b]$ implies $a(x) \neq a(y)$, which implies that there exists an $i$ which is the first integer such that $a_i(x) \neq a_i(y)$. This implies $h^{2i}(x) \in S_n$ and $h^{2i}(y) \in S_m$ with $m \neq n$.

For $x \in [a, b]$, the itinerary of $x$ under $h$ is

$$b(x) = (b_0(x), b_1(x), \ldots),$$

where $b_i(x) \in T$ for $i \geq 0$, $b_i(x) = T_n$ means $h^i(x) \in T_n$.

Consider the itineraries of $x$ and $y$ under $h^2$:

$$a(x) = (a_0(x), a_1(x), \ldots),$$
$$a(y) = (a_0(y), a_1(y), \ldots),$$

and $i$ is the first such integer such that $a_i(x) \neq a_i(y)$, which means for some $m \neq n$, $0 \leq m, n \leq 6$, $h^{2i}(x) \in S_n$ and $h^{2i}(y) \in S_m$. Now $S_n \subset T_n'$ for some $n'$, $0 \leq n' \leq 4$, and $S_m \subset T_{m'}$ for some $m'$, $0 \leq m' \leq 4$.

There are two cases:

1. If $T_n' \neq T_{m'}$, then the itineraries of $x$ and $y$ under $h$ are different.

2. Suppose $T_n' = T_{m'}$. Then $h^{2i}(x) \in S_n \subset T_n'$, $h^{2i}(y) \in S_m \subset T_{m'}$, and $T_n' = (a, c) = T_1$.

Therefore,

$$S_n, S_m \in \{(a, a_1), \{a_1\}, (a_1, c)\} = \{S_1, S_2, S_3\}.$$
(i) If $S_n = (a, a_1), S_m = (a_1, c)$, $h^{2i}(x) \in (a, a_1), h^{2i}(y) \in (a_1, c), h(a, a_1) = (c, b) = T_3$, and $h(a_1, c) = (a, c) = T_1$. Therefore $h^{2i+1}(x) \in T_3, h^{2i+1}(y) \in T_1$, which implies $b_{2i+1}(x) \neq b_{2i+1}(y)$, and the itineraries of $x$ and $y$ under $h$ are different.

(ii) If $S_n = \{a_1\}$ and $S_m = (a_1, c)$, then $h(\{a_1\}) = \{c\}$ and $h(a_1, c) = (a, c)$ imply $h^{2i+1}(x) \in T_2$ and $h^{2i}(y) \in T_1$, and the itineraries of $x$ and $y$ under $h$ are different.

(iii) If $S_n = \{a_1\}$ and $S_m = (a, a_1)$, then $h^{2i+1}(x) \in T_2$ and $h^{2i}(y) \in T_3$, and the itineraries of $x$ and $y$ under $h$ are different.

\[ \square \]

\textbf{Lemma 3.} Define $g : K \to K$ as follows: $g(a) = b, g(b) = \frac{a+b}{2}, g(\frac{a+b}{2}) = a$, $g$ is decreasing and linear on $[a, \frac{a+b}{2}]$ and $g$ is increasing and linear on $[\frac{a+b}{2}, b]$. Let $c = \frac{a+b}{2}$. The function $g$ is weakly expanding and transitive on $K$, and $g$ is conjugate to $h$ (where $h$ has the properties of the Lemma 1).

\textbf{Proof.} This follows from Theorem 12 [Baldwin (1990, Theorem 11)]. (We actually need the stronger form of Baldwin’s theorem here. Since we do not wish to go into Baldwin’s notation, we settle for referencing the original theorem.) Let $T = \{\{a\}, \{a, c\}, \{c\}, \{c, b\}, \{b\}\}$, and let $T_0 = \{a\}, T_1 = (a, c), T_2 = \{c\}, T_3 = (c, b), T_4 = \{b\}$ (so $T = \{T_0, T_1, T_2, T_3, T_4\}$). Then $T$ partitions $[a, b]$ into its endpoints and critical point ($T_0, T_2, T_4$); an open interval on which it is increasing; and an open interval on which it is decreasing. We can partition $[a, b]$ for $g$ in a similar manner: Let $T_g = \{\{a\}, (a, \frac{a+b}{2}), \{\frac{a+b}{2}, b\}, \{b\}\}$, and let $T_{g0} = \{a\}, T_{g1} = (a, \frac{a+b}{2}), T_{g2} = \{\frac{a+b}{2}, b\}, T_{g3} = (\frac{a+b}{2}, b), T_{g4} = \{b\}$ (so $T_g = \{T_{g0}, T_{g1}, T_{g2}, T_{g3}, T_{g4}\}$). Thus, Baldwin’s condition $E_h = E_g$ is satisfied, and $g$ is weakly expanding.

That $g$ is transitive follows Proposition 11, and the fact that, again, Baldwin’s condition $E_G = E_g$ is satisfied, so that $G$ (from Proposition 11) and $g$ are conjugate. \[ \square \]

The previous lemmas give us the following result:

\textbf{Theorem 13.} Suppose $h$ and $g$ have the properties of the lemma above. It follows that $h$ is weakly expanding on $K$, $h$ is conjugate to $g$, and we may completely understand the dynamics of $h$ on $K$ by considering the simpler piecewise linear map $g$. Furthermore, the orbit of some point in $K$ (under the action of $h$) is dense, and it must the case that

1. there is a residual set of points in $K$ each of which has an orbit dense in $K$,

2. the set of periodic points in $K$ is dense in $K$, and

3. $h$ is sensitive to initial conditions at each point of $K$.
Hence, \( h \) is chaotic in the sense of Devaney on \( K \). Then if \( Y = \lim (K, h) \), by Theorem 10, \( Y \) is an indecomposable continuum and by Theorem 11, \( Y \) is an arc continuum. (Thus \( Y \) is an indecomposable continuum, but it is rather simple for this class of continua in that it contains no indecomposable proper subcontinua.) Let \( H \) denote the homeomorphism induced by \( h \) on \( Y \), and let \( \sigma_H = H^{-1} \). It follows immediately that, under the action of \( \sigma_H \), the set of points in \( Y \) that have dense orbits in \( Y \) is a residual subset of \( Y \), the set of periodic points in \( Y \) form a dense subset of \( Y \), and \( \sigma_H \) is sensitive to initial conditions in \( Y \). In other words, \( \sigma_H \) is chaotic in the sense of Devaney on \( Y \).

Now suppose that the expansion on \([a, c]\) does not dominate the contraction on \((c, b)\): Suppose there is an interval \( I = [a, d] \) in \([a, c]\) such that \( h^3(I) \subset [a, d] \). (This might happen if, say, the max \( b \) of \( h \) corresponds to the local maximum value of the original (case I.A) \( h \) defined on \([a, c]\).) In this case the period three orbit \( \{a, b, c\} \) is attracting for an open set of points in the interval:

**Theorem 14.** Suppose \( f : [a, b] \to [a, b] \) is a type I.A backward map from the CIA model with \( f(a) = b \), \( f(b) = c \) (where \( a < c < b \)), and \( f(c) = a \) so \( \{a, c, b\} \) forms a period 3 cycle. Let \( m_1 = f'(a^+) := \lim_{x \uparrow a} |f'(x)| \), \( m_2 = f'(c^-) := \lim_{x \downarrow c} |f'(x)| \), and \( m_3 = f'(b^-) := \lim_{x \downarrow b} |f'(x)| \). If \( m_1m_2m_3 < 1 \), then \( \{a, c, b\} \) is an attracting period 3 point for \( f \). The basin \( B \) of attraction of \( \{a, c, b\} \) contains the largest intervals \( I_a := [a, a_1] \), \( I_b := (b_1, b] \) and \( I_c := (c_1, c_2) \) with \( a < a_1 < c_1 < c < c_2 < b_1 < b \) and \( \{a_1, c_1, b_1\} \) being another period three orbit for \( f \).

**Proof.** \( f|[a, c] \) is one-to-one and onto \([a, b]\) so there exists a unique \( \tilde{a} \in (a, c) \) such that \( f(\tilde{a}) = c \). \( f^2|[c, b] \) is one-to-one and onto \([a, b]\). \( f^3|[a, \tilde{a}] \) is one-to-one and onto \([a, b]\) with \( f^3(a) = a \) and \( f^3(b) = b \).

Note that for sufficiently small \( \epsilon > 0 \), \( f^3|[a, a + \epsilon] \) is \( C^1 \). Let \( f^3(a^+) := \lim_{x \uparrow a} f^3(x) \) (restricting \( x \in (a, a + \epsilon) \)). Note \( f^3(a^+) = m_1m_2m_3 < 1 \). so \( \{a\} \) is a stable fixed point of \( f^3 \) and \( \{a, c, b\} \) is an attracting period 3 point of \( f \). Let \( a_1 \in (a, \tilde{a}) \) be the smallest value in \((a, \tilde{a})\) such that \( f^3(a_1) = a_1 \). Since \( f^3(a^+) < 1 \) we have \( f^3(x) \) \( < \) \( x \) for all \( x \in (a, a_1) \) so \( [a, a_1] \) is in the basin of \( a \) under \( f^3 \). Let \( b_1 = f(a_1) \) and \( c_1 = f(b_1) = f^2(a_1) \). Note \( f(c_1) = f^3(a_1) = a_1 \) so \( \{a_1, c_1, b_1\} \) is a period three point under \( f \). Note \( a < a_1 < a_1 < c \) so \( c < f(a_1) \). \( b_1 < b \) and \( f(a_1) \neq a_1 \) (this implies \( a_1 \) is not the fixed point of \( f \)). Note that since \( a < a_1 < a_1 < c \) there are two points \( a_1 < c_1 < c < c_2 < b_1 \) such that \( f(c_1) = a_1 \) and \( f(a_1) = c_2 \). Clearly, \([a, a_1] \cup (c_1, c_2) \cup (b_1, b]\) is in \( B \).

The next theorem shows that in the case where the period three point is attracting, it is possible that the map \( f \) is chaotic only on a Cantor set:
Theorem 15. Suppose \( f : [a, b] \to [a, b] \) satisfies the properties of Theorem 14. Let \( m_1 := |f'(a^+)|, m_2 := |f'(c^-)|, m_3 := f'(b^-) \). \( m_4 := \inf_{x \in (c, b)} |f'(x)| \) and \( m_5 := \inf_{x \in [a, c]} |f'(x)| \) where \( a_1 \) is such that \([a, a_1] \) is the largest interval in the basin \( B \) of \([a, c, b]\) under \( f \). Again, for \( x_1 \in (a, c) \) being the unique solution to \( f(x_1) = c \), we let \( f'(x_1) \) be the set of values \( \{f'(x_1^+), f'(x_1^-)\} \). If \( m_1m_2m_3 < 1 \) and \( m_4m_5 > 1 \), then the basin \( B \) of attraction of \([a, c, b]\) is dense in \([a, b]\). Furthermore, \( C := [a, b] \setminus B \) is a Cantor set, \( f(C) = C \), and \( C \) contains periodic points of all periods.

Proof. First, we show next that \([a, b] \setminus B \) cannot contain an interval. Suppose to the contrary that \([a, b] \setminus B \) contains an interval \( K \). Then \( f^n(K) \cap B = \emptyset \) for all nonnegative integers \( n \). Then for each positive integer \( n \) we must have \( f^n(K) \subset (a_1, c_1) \) or \( f^n(K) \subset (c_2, b_1) \). Furthermore, if \( f^n(K) \subset (a_1, c_1) \) then \( f^{n+1}(K) \subset (c_2, b_1) \). It follows that the length of \( f^n(K) \) goes to infinity (because its length is bounded below by a product of \( m_4 \)'s, \( m_5 \)'s, and the length of \( K \)). Hence, \( C := [a, b] \setminus B \) is closed, nonempty and \( C \) contains no interval in \([a, b]\) so \( B \) is dense in \([a, b]\).

Next, we show that \( C \) is perfect. Since the only periodic orbit in \( B \) is the period 3 point \([a, c, b]\); by Sarkovskii’s theorem \( C \) must contain periodic points of all periods and so must be at least an infinite set. Suppose \( C \) is not perfect. Then there is a point \( x \in C \) and a closed interval \([r, s]\) containing \( x \) such that \([r, s] \cap C = \{x\} \). Then \([r, x] \cup (x, s] \subset B \), and there is some positive integer \( N \) such that if \( n > N \), \( \{f^n(r), f^n(s)\} \subset [a, a_1] \cup (c_1, c_2) \cup (b_1, b] \). Now \( f^n([r, s]) \) is an interval. If \( f^n([r, s]) \) is a subset of one of \([a, a_1]\), \((c_1, c_2]\), or \((b_1, b]\), then \( x \in B \), a contradiction. Then \( f^n([r, s]) \) is an interval in \([a, b]\) that intersects at least two of the sets \([a, a_1]\), \((c_1, c_2]\), and \((b_1, b]\). But again we have a contradiction - if, say, \( f^n([r, s]) \) intersects both \([a, a_1]\) and \((c_1, c_2]\), then the only point in \([a, c_1]\) not in \( B \) is \( f^n(x) \) and \( a_1 = c_1 = f^n(x) \). The other cases generate similar contradictions. Then \( C \) is perfect, and \( C \) is a Cantor set.

Corollary 1. Suppose \( f : I \to I \) satisfies the properties of Theorem 15. Then the induced homeomorphism \( F \) on the inverse limit set \( \underline{\lim}(I, f) \) has a period three orbit with a basin of attraction dense in \( \underline{\lim}(I, f) \) and there is an invariant cantor set \( \tilde{C} \subset \underline{\lim}(I, f) \) such that \( F|\tilde{C} \) is chaotic in the sense of Devaney. Furthermore, \( \sigma = F^{-1} \) has a period three orbit with a dense basin of repulsion and \( \sigma|\tilde{C} \) is chaotic in the sense of Devaney as well.

Example 2. There exists a type I.A backward map from the CIA model such that \( f(x) \) satisfies the hypotheses of Theorems 14 and 15. Let \( a < b \) and \( f : [a, b] \to [a, b] \) be given by
the following:

\[
f(x) := \begin{cases} 
m(x) & x \in [a, d] 
n(x) & x \in [d, e] 
p(x) & x \in [e, c_1] 
q(x) & x \in [c_1, e] 
r(x) & x \in [c, b] 
\end{cases}
\]  

where

\[
m(x) := b - \tau(x - a), 
n(x) := m(d) + n_1(x - d) + n_2(x - d)^2, 
p(x) := n(e) - \sigma_1(x - e), 
q(x) := p(c_1) - \sigma_2(x - e), 
r(x) := \gamma + (1 - \delta)x - \alpha x^2.
\]

\(n(x)\) is constructed and the parameters \(e\) and \(d\) chosen so that \(f\) is \(C^1\) on \([a, c_1]\) with \(f' < 0, f'' \leq 0\). This involves setting \(n_1 = -\tau\) and \(n_2 = (-\sigma_1 + \tau)/[2(e - d)]\).

Let \(a = 1, d = 1.09, e = 1.10, c = 23/16, b = 2.0, \tau = 0.1, \sigma_1 = 9, c_1 = (c - n(e))/(\sigma_1) + e, \) and \(\sigma_2 = (c - a)/(c - c_1)\). For these values we have \(m(d) = b - \tau(d - a) = 2 - 0.1(0.09) = 1.991\). The value for \(n(e)\) is given by \(n(e) = b - \tau(d - a) - (\sigma_1 + \tau)(e - d)/2\).

The largest interval \([a, a_1]\) in the basin \(B\) of the attracting period three point has \(a_1 = 1.10366971551532\) (with \(|f^3(a_1) - a_1| < 7e - 16\)). Then \(f(a_1) = b_1 = 1.37090125831213, f(b_1) = c_1 = 1.91247256036214, \) and \(a_1 = f(c_1)\) (within \(7e - 16\)). Note that \(e < a_1 < c_1\) so \(f'(a_1) = -\sigma_1\). Then \(m_1 = \tau, m_2 = \sigma_1, m_3 = f'(b^-) = 0.757, m_4 := \inf_{x \in (c, b)} f'(x) = 0.757, m_5 := \inf_{x \in (a_1, c)} |f'(x)| = \sigma_2\). Then we have \(m_1m_2m_3 = 0.1179580948508 < 1\) and \(m_4m_5 = 1.24183030243131 > 1\).

## 5.2 Simple Dynamics

The next theorem proves that even though the dynamics are not well-defined going forward in case I.A of the CIA model, the dynamics are relatively simple. In particular, we prove that the inverse limit space is an arc or a double topologist’s sin(1/x) curve in this case (which of these we get depends on how \([0, e]\) and \([d, 1]\) interact for the map under the action of \(f\)). The dynamics are therefore simple, even though forward in time the corresponding map from our family is not well-defined.

**Theorem 16.** Suppose \(0 < e < c < a < b < d < 1\), and suppose \(f : [0, 1] \to [0, 1]\) has the following properties:

(a) \(f([a, b]) = [e, d]\),
Figure 7: Backward map $f$ from equation (7). Parameters are such that the hypotheses of Theorem 15 are satisfied so the basin $B$ of the attracting period three point $\{a, c, b\}$ is dense in $[a, b]$ and the set $C := [a, b] \setminus B$ is a Cantor set containing periodic points of all orders.

(b) $f|[c, 1] : [c, 1] \rightarrow [0, 1]$ is one-to-one, onto, and decreasing,

c) $f|[0, e]$ is increasing, and

d) $f(c) = 1, f(0) = d = f(a), f(b) = e, f(1) = 0.$

Then $\lim_{n \rightarrow \infty}([0, 1], f)$ is either an arc or a double topologist’s $\sin(1/x)$ curve.

Proof. Note that by assumption, $[a, b] \subset (c, d)$, and $f|[a, b] : [a, b] \rightarrow [e, d]$ is one-to-one, onto, and decreasing. Let $Z = \lim_{n \rightarrow \infty}([0, 1], f)$. A point $\mathbf{x}$ in $Z$ can be represented as $\mathbf{x} = (x_0, x_1, x_2, \ldots)$.

(1) Let $f_0 = f|[a, b]$. Then $f_0^{-1} : [e, d] \rightarrow [a, b]$ is one-to-one and onto. Then $[a_1, b_1] := f_0^{-1}(a, b) \subset [a, b], [a_2, b_2] := f_0^{-1}(a_1, b_1) \subset [a_1, b_1]$, and, in general, $[a_n, b_n] = f_0^{-n}([a, b]) \subset [a_{n-1}, b_{n-1}]$. Let $X_0 = [e, d], X_1 = [a, b]$, and for $n > 1, X_n = [a_{n-1}, b_{n-1}]$. For each positive integer $n$, let $f_0|X_n = g_n$. Then $g_n : X_n \rightarrow X_{n-1}$ is a homeomorphism, and $\lim(X_n, g_n)$ is homeomorphic to an arc, which we call $Z_0$, and $Z_0 \subset \lim([0, 1], f) = Z$. Furthermore, if $\mathbf{x} \in Z$, and $x_0 \in [e, d]$, then $\mathbf{x} \in Z_0$. In fact, if $U_0 = \{\mathbf{x} \in Z : x_0 \in (e, d)\}$, then $U_0$ is open in $Z$, and $U_0 \subset Z_0$.

(2) Let $S = \{\mathbf{x} \in Z : x_i \in [e, d] \text{ and } x_{i+1} \in [a, b] \text{ for some nonnegative integer } i\}$. Then $Z_0 \subset S$, but $Z_0 \neq S$. In the argument that follows, we show that $S$ is the union of a countable collection $\{\ldots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \ldots\}$ of arcs having the property that $\cup_{i=-n}^{n} Z_i$ is an arc for each positive integer $n$. Because $f([0, a]) = [d, 1]$, and $f([b, 1]) = [0, e]$, there is a “flipping back and forth” of these intervals under the action of $f$. Hence, it is necessary to alternate the definitions of $Z_{2n}$ and $Z_{2n-1}$ to make sure that $Z_{2n-1} \cup Z_{2n}$ is the union of two arcs intersecting at one endpoint to form a larger arc.
(i) Let $Z_1 = \{x \in Z : x_1 \in [e, a], x_2 \in [a, b]\}$, and $Z_{-1} = \{x \in Z : x_1 \in [b, d], x_2 \in [a, b]\}$. Hence, $Z_1 \cup Z_{-1} \subseteq S$. Now $f|\{e, a\} : [e, a] \rightarrow [d, 1]$, and $f|[b, d] : [b, d] \rightarrow [0, e]$. Let $h_{11} = f|[e, a]$ and $h_{-11} = f|[b, d]$. Neither $h_{11}$ nor $h_{-11}$ is onto. Furthermore, $f^{-1}|\{e, a\}$ is one-to-one and $f^{-1}(\{e, a\}) \subset [a, b]$. In fact, for each $n > 0$, $f^{-n}(\{e, a\}) \subset [a, b]$ and $f|f^{-n}(\{e, a\}) : f^{-n}(\{e, a\}) \rightarrow f^{-n+1}(\{e, a\})$ is a homeomorphism. Let $X_{10} = [d, 1]$, $X_{11} = [e, a]$, and for $n > 1$, $X_{1n} = f^{-n+1}(\{e, a\})$. For $n > 1$, let $h_{1n} = f|X_{1n+1}$. Then $Z_1 = \lim (X_{1n}, h_{1n})$ is an arc, $Z_1 \subset Z$, and $Z_0 \cap Z_1 = \{(d, a, f^{-1}(a), f^{-2}(a), \ldots)\}$. Let $h_{12} = f|[a, f^{-1}(b)]$. Then $h_{12n} = f|\{b, d\}$ is one-to-one and $[a, f^{-1}(b)] = h_{12n}(\{b, d\}) \subset [a, b]$. Let $X_{-10} = [0, e]$, $X_{-11} = [b, d]$, $X_{-12} = [a, f^{-1}(b)]$, and for $n > 2$, $X_{-1n} = f^{-n+2}(X_{-12})$. For $n > 2$, let $h_{1n} = f|X_{1n+1}$. Then $Z_{-1} = \lim (X_{-1n}, h_{1n})$ is an arc, $Z_{-1} \subset Z$, $Z_0 \cap Z_{-1} = \{(e, b, f^{-1}(b), f^{-2}(b), \ldots)\}$, and $Z_{-1} \cap Z_1 = \emptyset$. Then $Z_{-1} \cup Z_0 \cup Z_1$ is an arc in $Z$ and in $S$.

(ii) This process continues: Next, let $Z_2 = \{x \in Z : x_2 \in [b, d], x_3 \in [a, b]\}$, and $Z_{-2} = \{x \in Z : x_2 \in [e, a], x_3 \in [a, b]\}$. By an argument similar to that above, $Z_2$ and $Z_{-2}$ are arcs, $Z_2 \cap Z_{-1} = \{(f(e), e, b, f^{-1}(b), f^{-2}(b), \ldots)\}$, $Z_{-2} \cap Z_{-1} = \{(f(d), d, a, f^{-1}(a), f^{-2}(a), \ldots)\}$, and $\cup_{i=2}^\infty Z_i$ is an arc. In general, if $n$ is a positive integer, $Z_{2n} = \{x \in Z : x_{2n} \in [b, d], x_{2n+1} \in [a, b]\}$, $Z_{2n-1} = \{x \in Z : x_{2n-1} \in [e, a], x_{2n} \in [a, b]\}$, $Z_{-2n} = \{x \in Z : x_{2n} \in [e, a], x_{2n+1} \in [a, b]\}$, and $Z_{(2n-1)} = \{x \in Z : x_{2n-1} \in [b, d], x_{2n} \in [a, b]\}$.

Thus, for each positive integer $m$, $Z_m$ and $Z_{-m}$ are arcs, and so is $\cup_{i=-m}^m Z_i$. Then $S = \cup_{i=-\infty}^\infty Z_i$ and $S$ is homeomorphic to the real line.

(iii) Let $S_1 = \{x \in Z :$ for each nonnegative integer $j$, $x_j \in [d, 1]$ if $j$ is odd, $x_j \in [0, e]$ if $j$ is even $\}$ and let $S_2 = \{x \in Z :$ for each nonnegative integer $j$, $x_j \in [d, 1]$ if $j$ is even, $x_j \in [0, e]$ if $j$ is odd $\}$. Note that $f(0, e) \subset [d, 1]$ and $f(d, 1) \subset [0, e]$. Hence, $[0, e] \supset f([d, 1]) \supset f^2([0, e]) \supset \cdots$, and $I_1 := \cap_{n=0}^\infty f^{2n}([0, e]) \neq \emptyset$ and $I_2 := \cap_{n=0}^\infty f^{2n}([d, 1]) \neq \emptyset$. If $x_0 \in I_1$, then $x_0 \in f^2(I_1)$, so there is $x_2 \in I_1$ such that $f^2(x_2) = x_0$. Since $x_2 \in I_1$, $x_2 \in f^2(I_1)$, and there is $x_4 \in I_1$ such that $f^2(x_4) = x_2$, and we can continue indefinitely. Thus, $(x_0, f(x_2), x_2, f(x_4), x_4, \ldots)$ is a point in $S_1$, so $S_1 \neq \emptyset$, and likewise $S_2 \neq \emptyset$. Since $f(I_1) = I_2$ and $f(I_2) = I_1$, either both $I_1$ and $I_2$ are intervals, or both are points. Since $f|[d, 1]$ is one-to-one and $f|[0, e]$ is one-to-one, $S_1$ is an arc if $I_1$ is an arc and $S_1$ is just one point if $I_1$ is just one point. (Both possibilities can occur.) If $S_1$ is an interval, $S_2$ is, too. If $S_1$ is a point, so is $S_2$.

Note that if a point $x$ is in $Z$, then either for each $n$, $x_n \in [0, e] \cup [d, 1]$, in which case $x$ is in $S_1$ or $x$ is in $S_2$; or, for some $n$, $x_n$ is in $(e, d)$, in which case $x$ is in $S$. Hence, $S$, $S_1$, and $S_2$ are mutually disjoint, and $S_1$ and $S_2$ are closed in $Z$. Thus, $S$ is open in $Z$. $S$ is also dense in $Z$: Note that $f^2((e, d)) = f((f(d), 1)) = [0, 1]$. Suppose $u$ is an nonempty open set in $Z$ of the form $U \cap Z$, where $U$ is a basic open set in the Hilbert cube in which $Z$ lies, i.e., for some $m$, $U = U_1 \times U_2 \times \cdots \times U_m \times [0, 1] \times [0, 1] \times \cdots$, with each $U_j$ open in $[0, 1]$. 38
Figure 8: A function on an interval that would yield simple dynamics and simple topology. The inverse limit for this map is an arc.

Then \( U' = U_1 \times U_2 \times \cdots \times U_m \times [0, 1] \times (e, d) \times [0, 1] \times [0, 1] \times \cdots \) is a nonempty open subset of \( U \) in the Hilbert cube, and \( U' \cap Z \neq \emptyset \). It follows that \( \emptyset \neq S \cap U' \subset S \cap U \), and that \( S \) is dense in \( Z \). The rest now follows easily: \( S_1 \) and \( S_2 \) are disjoint, nowhere dense, closed subsets of \( Z \), and \( S_1 \) and \( S_2 \) are contained in the closure of \( S \).

The next two examples give maps satisfying the hypotheses in Theorem 16. In the first case, the inverse limits is an arc, and for the second, the inverse limit is a double topologist’s \( \sin(1/x) \) curve.

**Example 3.** A map \( g \) satisfying the hypotheses of Theorem 16 exists such that the inverse limit space \( Z \) is an arc. Let \( f \) be the same form as that in Example 2 with parameters (notation from Example 2, not Theorem 16): \( a = 1 \), \( d = 1.10 \), \( e = 1.11 \), \( c = 1.4 \), \( b = 2.0 \), \( \tau = 1.0 \), \( \sigma_1 = 3.5 \), \( f(b) = 1.1 \), \( \gamma = 0.1627 \), \( \alpha = 0.2157 \), \( n(e) = 1.8775 \), The conjugate map (flipped and translated to \([0, 1]\)) is given by

\[
g(x) = 1 - [f(a + (b - a)(1 - x))] - a/(b - a).
\]

Figure 8 contains a plot of \( g \). Then \( g \) is continuous on \([0, 1]\) and satisfies the hypotheses of Theorem 16. Note that \( g^2([0, 0.2]) = g([0.9, 0.9161]) = [0.0839, 0.1] \subset [0, 0.2] \) and \( |g'(x)| < 0.2961 \) for \( x \in [0, 0.2] \). Also, \( g^2([0.9, 1]) = g([0, 0.1]) = [0.9, 0.9059] \subset [0.9, 1] \) and \( g'(x) = 1 \) for \( x \in [0.9, 1] \). We also have \( |g^2(x)| < 0.2961 \) for \( x \in [0, 0.2] \cup [0.9, 1] \). Hence, \( I_1 := \cap_{n=0}^{\infty} g^{2n}([0, 0.2]) \) consists of one point, as does \( I_2 := \cap_{n=0}^{\infty} g^{2n}([0.9, 1]) \neq \emptyset \).

**Example 4.** A map \( g \) satisfying the hypotheses of Theorem 16 exists such that the inverse limit space \( Z \) is a double topologist’s \( \sin(1/x) \) curve. Let \( f \) be the same form as that in
Figure 9: A function $g$ on an interval that would yield simple dynamics and simple topology. The inverse limit for this map is an double topologist’s sin($1/x$) curve. Also included is the second iterate of this map $g^2$ along with two invariant subsets $I_l := [x_1^l, x_2^l]$ and $I_h := [x_1^h, x_2^h]$.

Example 2 with parameters (again, using notation from Example 2, not Theorem 16): $a = 1$, $d = 1.05$, $e = 1.06$, $c = 1.4$, $b = 2.0$, $\tau = 0.5$, $\sigma_1 = 3.5$, $f(b) = 1.2$, $\gamma = 0.0667$, $\alpha = 0.1667$, $n(e) = 1.9550$, The conjugate map (flipped and translated to $[0,1]$) is given by

$$g(x) = 1 - \left[ f(a + (b-a)(1-x)) - a \right]/(b-a).$$

Figure 9 contains a plot of $g$. Then $g$ is continuous on $[0,1]$ and satisfies the hypotheses of Theorem 16.

The points $x_1^l = 0.01652657505897$ and $x_2^l = 0.52134397255332$ form a two cycle for $g^2$. The points $x_1^h = 0.80390172212762$ and $x_2^h = 0.96694684988205$ also form a two cycle for $g^2$. Let $I_l := [x_1^l, x_2^l]$ and $I_h := [x_1^h, x_2^h]$. In the figure, we see that $g^2|I_l$ is one-to-one and onto and $g^2|I_h$ is one-to-one and onto. Thus, $I_l = \cap_{n=0}^{\infty} g^{2n}(I_l)$ is an interval in this case. And $I_h := \cap_{n=0}^{\infty} g^{2n}(I_h)$ is an interval, too. Then the inverse limit space for this example is a double topologist’s sin($1/x$) curve.

6 Conclusion

In this paper, we demonstrate that the theory of inverse limits is the right tool for working with models that have ill-defined forward dynamics, but well-defined backward dynamics. The inverse limit approach is useful in at least two ways. First, the inverse limit can be useful for detecting or ruling out complex dynamics. This offers an alternative method that does not work directly with establishing the (non)existence of a 3-cycle (or other cycles and using Sarkovskii’s ordering on the integers). Second, we have shown that the backward map from the cash-in-advance model induces a homeomorphism on the inverse limit space.
This induced map on the inverse limit space topologically mimics the dynamics of the cash-in-advance model in the ill-defined directions, but has the enormous advantage of being well-defined moving forward and backward in time.

We have shown that the inverse limit spaces from the cash-in-advance model can behave quite differently, both topologically and dynamically. The topology and dynamics can be quite simple (a homeomorphism on an arc), or the space may be indecomposable - which automatically means the presence of chaotic behavior, although not necessarily on the entire continuum. There are many more questions one could ask and possibilities to explore, even about members of the family of maps we have studied, not to mention cases II.A and II.C (which we did not consider at all). Some of these possibilities include:

1. Ingram’s theorems (Theorems 7–11) can all be satisfied for maps conjugate to many members of the family of maps for our cash-in-advance model. Thus, the resulting inverse limit can be

   (a) a union of two Knaster bucket handle continua intersecting at a point or an arc
       (Theorem 7 is satisfied - the induced homeomorphism would flip one bucket handle
        onto the other);

   (b) an indecomposable continuum whether or not there is a period three point (Theorem
       9 is satisfied).

2. Ingram’s Theorem 8 gives a necessary and sufficient condition for the existence for a type (1) unimodal map of a periodic point \( p \) of odd period \( n \) greater than 1. If this theorem is satisfied, then the resulting inverse limit space would have to admit periodic points of all periods that succeed \( n \) in the Sarkovskii ordering. Thus, chaotic behavior is present on the entire inverse limit continuum or on at least a Cantor subset of that continuum.

3. If the map is a Markov map, then Ingram’s Theorem 11 gives sufficient conditions under which the resulting inverse limit space is an arc continuum.

4. Barge et al. (1996) prove that for a residual set of parameters \( (\lambda \in [1, 2]) \), the inverse limit space \( \lim_{\leftarrow}([0, 1], f_{\lambda}) \) is not only indecomposable, it also contains homeomorphic copies of every inverse limit of a tent map (that is, a copy of \( T_{\beta} \) for each \( \beta \in [1, 2] \)). (See subsection 3.2 for a definition of \( f_{\lambda} \).) Thus, not only is this space complex, it has “sublayers” that are themselves complex. This continuum must contain uncountably many topologically different indecomposable proper subcontinua. We do not know if
this is true for at least some member of our family of models, but we conjecture that it is.

In additions to exploring these possibilities, in future research we plan to use the notion of an invariant measure of a dynamical system to describe the ergodic distribution(s) of chaotic equilibria in the CIA model. This might be useful in “ranking” chaotic equilibria and providing a metric for policy evaluation. This is particularly useful in cases where the entire class of policies under consideration allows for chaotic equilibria. We also believe that the invariant measure can also be used to induce a measure on the inverse limit providing answers to the following important questions: (1) what is the measure the set of initial conditions that may lead to complex dynamics? and (2) if an equilibrium were picked at random what is the probability it would be chaotic?
References


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