Chaotic Equilibria in Models with Ill-Defined Forward Dynamics

Judy Kennedy* David R. Stockman†

January 2006

Abstract

Some economic models like the cash-in-advance model of money or overlapping generations model have the property that the dynamics are ill-defined going forward in time, but well-defined going backward in time. In such instances, what does it mean for an ill-defined dynamical system to be chaotic? Furthermore, under what conditions are such dynamical systems chaotic? In this paper, we provide a definition of chaotic that is in the spirit of Devaney for a dynamical system with ill-defined forward dynamics. We utilize the theory of inverse limits to provide necessary and sufficient conditions for such a dynamical system to be chaotic.

Keywords: cash-in-advance, overlapping generations, chaos, inverse limits.

JEL: C6, E3, and E4.

*Department of Mathematical Science, University of Delaware, Newark, DE 19716.
†Department of Economics, University of Delaware, Newark, DE 19716.
1 Introduction

In this paper we offer a definition of chaotic for dynamical systems that are ill-defined going forward in time but well-defined going backward in time and provide necessary and sufficient conditions for such a dynamical system to be chaotic. The term chaotic when applied to dynamical systems has been defined in several non-equivalent ways.\(^1\) Our definition of chaotic is in the spirit of one of the more commonly used definitions of chaotic, namely that of Devaney (2003). Our results on necessary and sufficient conditions for chaos utilize the theory of inverse limits to establish that there is chaos for the well-defined backward dynamical system if and only if there is chaos for the ill-defined forward dynamical system.\(^2\)

The equilibrium of a dynamic economic model can often be characterized as a solution to a dynamical system. Many nonlinear dynamical systems are well-defined moving forward, but not well-defined going backward. However, in economics there are dynamical systems with the opposite property, namely dynamics that are not well-defined going forward, but are well-defined going backward. Two such models include the overlapping generations (OLG) model and the cash-in-advance (CIA) model.\(^3\) Typically, the problem of ill-defined forward dynamics is either ignored by using a local analysis or avoided by analyzing the model with the well-defined backward map as in Grandmont (1985) and Michener and Ravikumar (1998).\(^4\) However, using a local analysis ignores some potentially interesting equilibria. And as Medio (1992, pp. 222–23) has noted, the backward map solution is not entirely satisfactory either because the backward map gives trajectories that go backward into the infinite past, whereas equilibria are trajectories that lead off into the infinite future.

The inverse limit of a dynamical system is a subset of an infinite dimensional space (e.g. the Hilbert cube) where each point in the inverse limit corresponds to a backward solution (backward orbit) of the dynamical system. Using the backward map from say the CIA model as our dynamical system, a point in the inverse limit, being a backward orbit of the backward map, corresponds to a forward orbit in the model. The well-defined backward map can be used to induce a homeomorphism on the inverse limit space. The dynamical properties of this induced homeomorphism are closely related to those of the well-defined backward map.\(^5\)

\(^1\)See, for example Robinson (1995, pp. 83–84).
\(^2\)Inverse limits have been used to investigate the ill-defined forward dynamics in economic models by Medio and Raines (2005a,b) and Kennedy et al. (2004, 2005).
\(^3\)See Grandmont (1985) for the OLG model and Michener and Ravikumar (1998) for the CIA model.
\(^4\)Grandmont (1985) also utilizes a “belief function” describing how agents form their beliefs about the future to the remedy problem of ill-defined forward dynamics.
\(^5\)The two maps are typically not conjugate (\(F\) is a homeomorphism and \(f\) need not be), but they are what is called \(\epsilon\)-conjugate.
Surprisingly, the dynamical properties of the inverse of this homeomorphism (a well-defined function) are closely related to those of the forward dynamical system which is not even well-defined. Though our motivation for this problem stems from the CIA and OLG models where the state space is an interval of the real line, our results are much more general and hold for any dynamical system where the state space is a metric space.

The paper is organized as follows. To motivate our problem, in section 2 we briefly cover the cash-in-advance model to illustrate that the implicitly-defined difference equation characterizing equilibria is well-defined going backward in time, but not well-defined going forward in time. In section 3 we discuss inverse limits and the induced homeomorphism on this space by the dynamical system. We also define chaotic for ill-defined dynamical systems and provide necessary and sufficient conditions for such a dynamical system to be chaotic. We conclude in section 4.

2 The Model

The model is the standard endowment CIA model of Lucas and Stokey (1987). We closely follow the exposition of Michener and Ravikumar (1998), hereafter [MR]. It is an endowment economy with both cash and credit goods. There is a representative agent and a government. The government consumes nothing and sets monetary policy using a money growth rule.

The household has preferences over sequences of the cash good \( c_{1t} \) and credit good \( c_{2t} \) represented by a utility function of the form

\[
\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}),
\]

with the discount factor \( 0 < \beta < 1 \). To purchase the cash good \( c_{1t} \) at time \( t \) the household must have cash \( m_t \). This cash is carried forward from \( t - 1 \) and in this sense the household is required to have cash in advance of purchasing the cash good. The credit good \( c_{2t} \) does not require cash, but can be bought on credit. The household has an endowment \( y \) each period that can be transformed into the cash and credit goods according to \( c_{1t} + c_{2t} = y \). Since this technology allows the cash good to be substituted for the credit good one-for-one, both goods must sell for the same price \( p_t \) in equilibrium and the endowment must be worth this price per unit as well. Each period the household also receives a transfer of cash from the government in the amount \( \theta M_t \).

The household seeks to maximize (1) by choice of \( \{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty} \) subject to the con-
straints $c_{1t}, c_{2t}, m_{t+1} \geq 0,$

\begin{align}
    p_t c_{1t} & \leq m_t, \quad \text{(2)} \\
    m_{t+1} & \leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t}, \quad \text{(3)}
\end{align}

taking as given $m_0$ and $\{p_t, M_t\}_{t=0}^\infty$. The money supply $\{M_t\}$ is controlled by the government and follows a constant growth path $M_{t+1} = (1 + \theta)M_t$ where $\theta$ is the growth rate and $M_0 > 0$ given.

A perfect foresight equilibrium is defined in the usual way as a collection of sequences $\{c_{1t}, c_{2t}, m_t\}_{t=0}^\infty$ and $\{M_t, p_t\}_{t=0}^\infty$ satisfying the following.

1. The money supply follows the stated policy rule: $M_{t+1} = (1 + \theta)M_t$.
2. Markets clear: $m_t = M_t$ and $c_{1t} + c_{2t} = y$.
3. The solution to the household optimization problem is given by $\{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^\infty$.

[MR] make assumptions on the function $U$ so that the solution to this problem will be interior and the solution to the first-order conditions and transversality condition will be necessary and sufficient.

**Assumption 1.** ([MR], p. 1120) The function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is $C^2$ with $U_1 > 0, U_2 > 0$ and the Hessian matrix negative definite. Both $c_{1t}$ and $c_{2t}$ are assumed to be normal goods. Further, to guarantee interior solutions we will assume

$$\lim_{c \to 0} U_1(c, y-c) = \lim_{c \to y} U_2(c, y-c) = \infty,$$

and that $U_1(y, 0) < \infty$ and $U_2(0, y) < \infty$.

The first-order conditions from the household’s problem imply that

$$U_2(c_{1t}, c_{2t})/p_t = \beta U_1(c_{1t+1}, c_{2t+1})/p_{t+1}. \quad \text{(4)}$$

This condition reflects that at the optimum, the household must be indifferent between spending a little more on the credit good (giving a marginal benefit $U_2(c_{1t}, c_{2t})/p_t$) versus savings the money and purchasing the cash good in the next period (giving a marginal benefit $\beta U_2(c_{1t+1}, c_{2t+1})/p_{t+1}$).

Let $x_t := m_t/p_t$ denote the level of real money balances. Using the equilibrium conditions that $M_t = m_t$ and $c_{2t} = y - c_{1t}$, equation (4) implies

$$x_t U_2(c_{1t}, y - c_{1t}) = \frac{\beta}{1 + \theta} x_{t+1} U_1(c_{1t+1}, y - c_{1t+1}). \quad \text{(5)}$$
If the cash-in-advance constraint (2) binds, then \( c_{1t} = x_t \). If not, then the Lagrange multiplier \( \mu_t = 0 \) and \( c_{1t} = c := \arg\max_x U(x, y - x) \). It then follows that \( c_{1t} = \min[x_t, c] \) for all \( t \). Using this relationship we can eliminate \( c_{1t} \) and \( c_{1t+1} \) from (5) to get a difference equation in \( x \) alone:

\[
x_t U_2(\min[x_t, c], y - \min[x_t, c]) = \frac{\beta}{1 + \theta} x_{t+1} U_1(\min[x_{t+1}, c], y - \min[x_{t+1}, c])
\]

or

\[
B(x_t) = A(x_{t+1}),
\]

where

\[
B(x) := x U_2(\min[x, c], y - \min[x, c]),
\]

\[
A(x) := \frac{\beta}{1 + \theta} x U_1(\min[x, c], y - \min[x, c]).
\]

Whether or not the dynamics going forward are well-defined depends on whether or not \( A(\cdot) \) is invertible.

[MR] use two more assumptions in their paper which we include here for completion and briefly describe what they imply for the model.

Assumption 2. ([MR], p. 1125) There exists a \( b \in [0, c) \) such that \( xU_1(x, y - x) \) is increasing in the region \([0, b)\) and decreasing in the region \((b, c]\).

This assumption is putting additional restrictions on the utility function so that the function \( A(\cdot) \) is either hump-shaped or monotonically decreasing on \([0, c]\).

Assumption 3. ([MR], p. 1125) (a) \((1 + \theta) > \beta\) and (b) \( b < x^* \).

These conditions guarantee the existence of a solution \( x^* > 0 \) to \( A(x^*) = B(x^*) \) and that this intersection of the two functions occurs when \( A(x) \) is decreasing.

Properties of the Implicit Difference Equation

Consider the difference equation defined by (6) from above and recall that we are interested in the solutions to the difference equation, which are sequences \( x_0, x_1, x_2, \ldots \) of nonnegative real numbers satisfying the difference equation. Under assumptions 1–3, the functions \( A \) and \( B \) are continuous functions from \([0, \infty)\) to \([0, \infty)\) with the following properties:

1. \( B \) is increasing and therefore one-to-one, but \( A \) is not one-to-one.
2. For some positive number $c$, both $A$ and $B$ are linear on $[c, \infty)$ with positive slopes, and the slope of $A|[c, \infty)$ is less than the slope of $B|[c, \infty)$.

3. On some interval $[0, b]$, (with $b < c$) the behavior of $A$ may be increasing with $A(0) = 0$ (type I), or it may be increasing with $A(0) > 0$ (type II), or $A$ may be decreasing on $[0, c]$ (type III). For type III we let $b = 0$. See Figure 1 for illustrations of type I and III configurations for $A$ and $B$. (Type II looks similar to type I, but with $A(0) > 0$.)

4. On the interval $(b, c]$, $A$ is decreasing, with $x^* \in (b, c)$ such that $A(x^*) = B(x^*)$.

Figure 1: Type I (left) and III (right) configurations for $A$ and $B$.

Note that there are positive numbers $\underline{x}$ and $\overline{x}$ such that

\[ B(\underline{x}) = A(c), \]
\[ B(\overline{x}) = A(\underline{x}), \]

and in type I and possibly type II there are positive numbers $\underline{x}^b$ and $\overline{x}^b$ such that

\[ B(\underline{x}^b) = A(b), \]
\[ B(\overline{x}^b) = A(\overline{x}^b). \]

Since $A$ is not one-to-one, the dynamics in the model given by the difference equation (6) are not well-defined going forward in time. However, since $B$ is one-to-one, we can invert $B$ and define the backward map $f(x) := B^{-1} \circ A(x)$. This function gives the backward dynamics
$x_t = f(x_{t+1})$, maps $[0, \infty)$ to itself and inherits the basic shape of $A$. Consequently, even though the dynamics of (6) are not well-defined going forward in time, the dynamics are well-defined going backward in time. In terms of the $f$ function we have: $x = f(c)$, $\overline{x} = f(\overline{x})$, $\overline{x}^b = f(b)$, and $\underline{x}^b = f(\underline{x})$. Note that if $b > 0$, we have $A(x) \leq A(b)$, which implies $B(\overline{x}) = A(\overline{x}) \leq A(b) = B(\overline{x})$. Since $B$ is increasing, this implies $\overline{x} \leq \overline{x}^b$. Also, if $b > 0$, we have $A(b) \geq A(c)$ (since $\overline{x}^b > b$), which implies $B(x) = A(c) \leq A(\overline{x}^b) = B(\overline{x}^b)$. Since $B$ is increasing, this implies $x \leq x^b$. [MR] provide the following propositions about the attracting sets for the backward map $f$.

**Proposition 1 ([MR], Lemma 2, p. 1125).** If $0 < b \leq \overline{x}$, then the attracting set for $f$ is $J := [x, \overline{x}]$, i.e., for any solution $\ldots, x_{-2}, x_{-1}, x_0$ with $x_0 > 0$ to $x_{t-1} = f(x_t)$ there exists some negative integer $T$ such that $x_t \in J$ for all $t \leq T$.

With $0 < b \leq \overline{x}$ and $J := [x, \overline{x}]$, there are three generic possibilities for $f$|J (see Figure 2):

I.A. $\overline{x} > c$,
I.B. $\overline{x} = c$,
I.C. $\overline{x} < c$.

Figure 2: The figure illustrates the generic possible shapes for $f : J \rightarrow J$ in each of cases I.A–I.C when $0 < b \leq \overline{x}$.

**Proposition 2 ([MR], Lemmas 3 and 4, pp. 1126–27).** Under assumptions 1–3, if $\overline{x} < b$ the attracting set for $f$ depends on the relative magnitudes of $c$ and $\overline{x}^b$. If $\overline{x}^b \leq c$, then the attracting set for $f$ is $J := [x^n, \overline{x}^b]$, i.e., for any solution $\ldots, x_{-2}, x_{-1}, x_0$ with $x_0 > 0$ to $x_{t-1} = f(x_t)$ there exists some negative integer $T$ such that $x_t \in J$ for all $t \leq T$. If $\overline{x}^b > c$, then the attracting set for $f$ is $J := [x, \overline{x}^b]$, i.e., for any solution $\ldots, x_{-2}, x_{-1}, x_0$ with $x_0 > 0$ to $x_{t-1} = f(x_t)$ there exist some negative integer $T$ such that $x_t \in J$ for all $t \leq T$. 

7
With $b > x$, there are four generic possibilities for $f|J$ (see Figure 3):

II.A. $x \leq x^b < b < x^b < c$ with $J := [x^b, x^b]$,

II.B. $x < b = x^b < x^b < c$ with $J := [x^b, x^b]$,

II.C. $x \leq x^b < b < x^b < c$, with $J := [x^b, x^b]$,

II.D. $x < b < c < x^b$ with $J := [x, x^b]$.

Figure 3: The figure illustrates the generic possible states for $f|J : J \rightarrow J$ in each of the cases II.A–II.D when $x < b$.

When considering forward solutions to (6), one can argue that if $x_t$ ever leaves a certain range the behavior is relatively simple. The forward solution (if it exists) will eventually become monotonic with $x_t \rightarrow \infty$ or $x_t \rightarrow 0$. These possibilities about the forward solutions to the the difference equation (6) are precisely described in the following propositions from Kennedy et al. (2004).

**Proposition 3.** For cases I.A–I.C and II.D, if $(x_0, x_1, \ldots)$ is a solution to $A(x_{t+1}) = B(x_t)$ such that $x_{\hat{t}} < x$ for some $\hat{t}$, then

[a] for $t \geq \hat{t}$, the choice of $x_{t+1}$ is unique, i.e., $x_{t+1}$ such that $A(x_{t+1}) = B(x_t)$ is unique;

[b] $\lim_{t \rightarrow \infty} x_t = 0$; and

[c] $x_{\hat{t}} > x_{\hat{t}+1} > x_{\hat{t}+2} > \cdots$.

**Proposition 4.** For case I.A–I.C, if $(x_0, x_1, \ldots)$ is a solution to $A(x_{t+1}) = B(x_t)$ such that $x_{\hat{t}} > \bar{x}$ for some $\hat{t}$, then the choice of $x_{t+1}$ may not be unique, but either

[a] $\lim_{t \rightarrow \infty} x_t = \infty$ and eventually $x_t < x_{t+1} < x_{t+2} < \cdots$, or

[b] $\lim_{t \rightarrow \infty} x_t = 0$ and eventually $x_t > x_{t+1} > x_{t+2} > \cdots$. 

8
Proposition 5. For cases II.A–II.D, if \((x_0, x_1, \ldots)\) is a solution to \(A(x_{t+1}) = B(x_t)\) such that \(x_{\hat{t}} > x^b\) for some \(\hat{t}\), then

[a] \(t \geq \hat{t}\), the choice of \(x_{t+1}\) is unique, i.e., \(x_{t+1}\) such that \(A(x_{t+1}) = B(x_t)\) is unique;

[b] \(\lim_{t \to \infty} x_t = \infty\); and

[c] \(x_{\hat{t}} < x_{\hat{t}+1} < x_{\hat{t}+2} > \cdots\).

Proposition 6. For case II.A–II.C, if \((x_0, x_1, \ldots)\) is a solution to \(A(x_{t+1}) = B(x_t)\) such that \(x_{\hat{t}} < x^b\) for some \(\hat{t}\), then the choice of \(x_{\hat{t}+1}\) may not be unique, but either

[a] \(\lim_{t \to \infty} x_t = \infty\) and eventually \(x_t < x_{t+1} < x_{t+2} < \cdots\), or

[b] \(\lim_{t \to \infty} x_t = 0\) and eventually \(x_t > x_{t+1} > x_{t+2} > \cdots\).

In summary, Propositions 3 and 4 imply that for cases I.A–I.C, any forward solution that leaves \([\underline{x}, \overline{x}]\) is not interesting. Propositions 3 and 5 imply that for case II.D, any forward solution that leaves \([\underline{x}, \overline{x}]\) is not interesting. And finally, Propositions 5 and 6 imply that for cases II.A–II.C, any forward solution that leaves \([\underline{x}^b, \overline{x}^b]\) is not interesting.

For cases I.A, I.B, II.A, II.B, and II.D, \(f|J : J \to J\) is onto, so if \(x_t \in J\) there is a point \(x_{t+1} \in J\) such that \(x_t = f(x_{t+1})\). However, in in cases I.C and II.C, the map \(f|J : J \to J\) is not onto, this implies that moving forward in time, some points must be thrown out of \(J\) implying uninteresting dynamics. Since we are interested in potentially interesting dynamics, we remove these points from \(J\). Let \(K\) be the the collection of points not removed (this is nonempty since the steady state solution \(x^* \in J\)). Since \(f|J\) is monotonic, one can show that \(K\) must be of the form \(K := [\underline{z}, \overline{z}] \subset J\) with \(\underline{z} \leq \overline{z}\). Note that that \(f|K : K \to K\) is onto and the picture looks similar to I.B or II.B if \(\underline{z} < \overline{z}\). However, it is possible for \(\underline{z} = \overline{z}\), in this case the only bounded solution is \(x_t = x^*\). To simplify notation, we will denote \(K\) by \(J\) and always consider \(f|J : J \to J\). The important structure to note is that \(J\) is a compact metric space and \(f|J\) may not be invertible, i.e., the dynamics are ill-defined going forward in time.

It follows from the previous propositions that in cases I.A–I.C and II.A–II.D solutions that contain members not in the interval \(J\) exhibit simple behavior. A solution containing a member not in \(J\) would be locked into one behavior - either its members would eventually increase without bound, or they would eventually decrease to 0. Such solutions may or may not constitute an equilibrium (the transversality condition may be violated).6

6See Woodford (1994) for a careful discussion of these cases.
3 Inverse Limits and Chaos for Dynamical Systems with Ill-Defined Forward Dynamics

In this section, we define chaotic for a dynamical system that is ill-defined going forward in time and provide some background on the uses of the term chaotic in the dynamical systems literature. We use the theory of inverse limits to show that the ill-defined dynamical system is chaotic if and only if the well-defined backward map from this system is chaotic.

3.1 Inverse Limits

Let $X$ be a nonempty compact metric space and suppose $f : X \rightarrow X$ is a continuous function. Let $Q = X^\infty$ be the infinite product of $X$ with itself endowed with the usual product topology. What is this topology, and how can we define a metric on $Q$ related to the original metric on $X$? Given a metric space $X$ with metric $d$, we can assume without loss of generality that $d$ is a bounded metric. (To see this, suppose $d$ is our original metric on $X$. Define $d'$ on $X \times X$ as follows: for $(x, y) \in X \times X$, let $d'(x, y) = d(x, y)$ if $d(x, y) \leq 1$; otherwise $d'(x, y) = 1$. It is easy to check that $d'$ is indeed a metric on $X$ bounded above by 1, and that $d'$ is compatible with the topology generated by the original $d$.)

Now for the product topology: Let $\{u_0, u_1, \ldots, u_n\}$ be a finite collection of open sets in $X$. We define $< u_0, u_1, \ldots u_n > := \{x = (x_0, x_1, \ldots) \in X^\infty : x_i \in u_i \text{ for } 0 \leq i \leq n\}$. The collection $B := \{< u_0, u_1, \ldots u_n > : \{u_0, u_1, \ldots u_n\} \text{ is a finite collection of open sets in } X\}$ is the collection of basic open sets. This set forms a basis for the usual product topology on $X^\infty$. For a metric on $X^\infty$ we will use the following function $\tilde{d} : X^\infty \rightarrow \mathbb{R}_+$ given by

$$\tilde{d}(x, y) := \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$  

The product topology is compatible with the topology generated by the metric $\tilde{d}$. Note the following standard facts from topology: (1) If $X$ is a metric space, then $X^\infty$ is a metric space. (2) If $X$ is compact, then $X^\infty$ is compact. (3) If $X$ is connected, then $X^\infty$ is connected. (4) If $X$ is a compact, connected metric space (also called a continuum), then $X^\infty$ is a compact, connected metric space, or a continuum. (5) If $X$ is a continuum, then $X^\infty$ is an infinite-dimensional space. (6) If $X$ is the unit interval, then $X^\infty$ is a copy of the familiar Hilbert cube. (7) If $X$ is homeomorphic to the $n$-dimensional cube $I^n$ (for some positive integer $n$ and $I$ an interval), then $X^\infty$ is a copy of the Hilbert cube.

Let $\mathbb{Z}^+$ denote the non-negative integers. The space $X$ is called the factor space and the function $f$ is called the bonding map. The pair $(X, f)$ is called an inverse system. The set
of points
\[ \lim(X, f) := \{ x = (x_0, x_1, \ldots) \in Q \mid x_i = f(x_{i+1}) \text{ for } i \in \mathbb{Z}^+ \}, \]
is the inverse limit of the inverse system \((X, f)\). Note that \(\lim(X, f)\) is a subset of \(X^\infty\) and each point in the inverse limit corresponds to a backward solution to the dynamical system \(f : X \to X\).\(^7\) Note that if \(f\) is the backward map from an economic model, then points in the inverse limit space correspond to forward solutions of the implicit difference equation characterizing an equilibrium in the model (i.e., backward orbits of the backward map).

Let \(Z := \lim(X, f)\). A natural map is induced on the inverse limit space by the bonding map \(f\): for \(x = (x_0, x_1, \ldots) \in Z\), define
\[ F(x) \equiv F((x_0, x_1, \ldots)) := (f(x_0), f(x_1), \ldots) \equiv (f(x_0), x_0, x_1, \ldots). \]
The induced map \(F\) is a homeomorphism from \(Z\) onto \(Z\). The inverse \(\sigma := F^{-1}\) of \(F\), is then defined by \(\sigma(x) \equiv \sigma((x_0, x_1, \ldots)) := (x_1, x_2, \ldots)\). Thus, the pair \((Z, F)\) forms a dynamical system, one that runs both forward and backward. The induced map \(\sigma\) is called the shift homeomorphism. If \(m \in \mathbb{Z}^+\), the map \(\pi_m : \lim(X, f) \to X\) defined by \(\pi_m(x) = x_m\) is called the projection map (or, if specificity is required the \(m^{\text{th}}\) projection map).

3.2 Chaos for Ill-Defined Maps
The only requirement for chaos, as defined originally by Li and Yorke (1975) was sensitivity to initial conditions. Suppose \(X\) is a metric space and \(f : X \to X\) is a map. Then \(f\) has sensitive dependence on initial conditions if there exists a sensitivity constant \(\delta > 0\) such that, for any \(x \in X\) and any neighborhood \(N\) of \(x\), there exists \(y \in N\) and an integer \(n \geq 0\) such that \(d(f^n(x), f^n(y)) > \delta\). While there are a number of different definitions of chaos in use today, most mathematicians require a condition known as “transitivity” to be satisfied. Here again the exact meaning of this word varies according to the mathematician using it.

Suppose \(X\) is a compact metric space and \(f : X \to X\) is a continuous function. We will say that \(f\) is transitive if whenever \(U\) and \(V\) are open sets (non-empty), there exists a positive integer \(n\) such that \(f^n(U) \cap V \neq \emptyset\).\(^8\) For \(x \in X\), the orbit of \(x\) under the action of \(f\) is

\(^7\)More generally, one has a sequence of factor spaces \(\{X_1, X_2, \ldots\}\) where \(X_i\) is a nonempty metric space, and a sequence of bonding maps \(\{f_1, f_2, \ldots\}\) such that \(f_i : X_{i+1} \to X_i\) for \(i \in \mathbb{Z}^+\). In this case, the collection \((X_m, f_m)\) is an inverse system, and the inverse limit is defined in an analogous way. So more carefully, we would say that our inverse system \((X, f)\) should be written as \((X_m, f_m)\) where \(X_m = X\) and \(f_m = f\) for all \(m \in \mathbb{Z}^+\).

\(^8\)Other terminology includes one-sided transitivity and topological transitivity. Sometimes these concepts are equivalent, but not always. See Walters (1982) for more on transitivity.
defined by \( O^f_+(x) := \{ x, f(x), f^2(x), \ldots \} \). Some authors say that \( f \) is transitive if there exists an \( x \in X \) such that \( O^f_+(x) \) is dense in \( X \).

The next theorem establishes that \( f \) is transitive on \( X \) iff the induced homeomorphism \( F \) is transitive on the inverse limit space \( \lim(X, f) \) as well.

**Theorem 1.** Suppose \( X \) is a compact metric space, and \( f : X \to X \) is continuous, \( Z = \lim(X, f) \), and \( F \) denotes the homeomorphism on \( Z \) induced by the map \( f \). Then \( f \) is transitive on \( X \) if and only if \( F \) is transitive on \( Z \).

**Proof.** Suppose \( u = < u_0, u_1, \ldots u_n > \) and \( v = < v_0, v_1, \ldots v_m > \) are basic open sets in \( X^\infty \). Without loss of generality assume \( n = m \). Suppose that \( u \cap Z \neq \emptyset \) and \( v \cap Z \neq \emptyset \). Then \( u \cap Z \) and \( v \cap Z \) are nonempty open sets in \( Z \). Then there is some \( x = (x_0, x_1, x_2, \ldots) \in u \cap Z \), and since, for each \( j \), \( 0 \leq j \leq n \), \( f^j(x_n) \in u_{n-j} \), we have \( x_n \in f^{-j}(u_{n-j}) \). Thus, \( \hat{u} := \cap_{j=0}^n f^{-j}(u_{n-j}) \neq \emptyset \) and is open in \( X \). Likewise, \( \hat{v} := \cap_{j=0}^n f^{-j}(v_{n-j}) \neq \emptyset \) and is open in \( X \). Since \( f \) is transitive on \( X \), there is some positive integer \( p \) such that \( f^p(\hat{u}) \cap \hat{v} \neq \emptyset \). Hence, \( \hat{u} \cap f^{-p}(\hat{v}) \) is open and nonempty in \( X \). Suppose \( y \in \hat{u} \cap f^{-p}(\hat{v}) \). Let \( k = n + p \). For \( 0 \leq i \leq k \), let \( f^i(y) = z_{k-i} \). There is some point \( w = (w_0, w_1, \ldots) \in Z \) such that \( f^{-i}(w_i) = z_i \). We have \( w_i = v_i \) for \( i = 0, 1, 2, \ldots n \). To see this note that \( w_n = f^p(y) \in \hat{v} \subset v_n \) and \( w_i = v_i \) for \( i = 1, 2, \ldots (n-1) \) since \( w_{i-1} = f(w_i) \). This implies that \( w \in v \cap X \). Essentially hitting \( y \) with \( f \) \( p \)-times puts the \( f^p(y) \) in \( \hat{v} \) and the remaining \( n \)-iterations travel backward through the \( v_i \)’s. We also have \( w_{p+i} \in u_i \) for \( i = 1, 2, \ldots, n \) so using the shift map \( p \)-times puts \( w \) in \( u \cap X \), i.e., \( \sigma^p(w) \in u \cap X \). Since the shift map is the inverse of the map \( F \) we have \( w \in F^p(u \cap X) \). So we have \( w \in (v \cap X) \cap F^p(u \cap X) \). Thus, \( F \) is transitive on \( Z \).

Suppose \( F \) is transitive on \( Z \). Let \( u, v \) be non-empty open sets in \( X \) and consider the open sets in \( Z \) given by \( \hat{u} := < u > \) and \( \hat{v} := < v > \). Since \( F \) is transitive there exists a non-negative integer \( p \) such \( F^p(\hat{u}) \cap \hat{v} \neq \emptyset \). Since \( F^p(\hat{u}) = \{ f^p(u), f^{p-1}(u), \ldots, f(u), u, X, X, \ldots \} \), we have \( f^p(u) \cap v \neq \emptyset \) so \( f \) is transitive. \( \square \)

We now give meaning to \( f^{-1} \) being transitive and having sensitive dependence on initial conditions, whether \( f^{-1} \) is well defined or not. Suppose \( X \) is a metric space and \( f : X \to X \) is continuous. Whether or not \( f^{-1} \) is well-defined, we can identify orbit(s) \( O^f_{-1}(x) \) of the point \( x \) under the action of \( f^{-1} \) with points in the inverse limit space \( Z := \lim(X, f) \):

\[
O^f_{-1}(x) := \{ x \in Z | \pi_0(x) = x \}.
\]

Then \( f^{-1} \) has sensitive dependence on initial conditions if there exists a sensitivity constant \( \delta > 0 \) such that, for any \( x \in X \) and any neighborhood \( N \) of \( x \), there exists \( y \in N \) and an
integer \( n \geq 0 \) such that \( d(x_n, y_n) > \delta \) for some \( x_n \in \pi_n(O_{+}^{-1}(x)) \) and \( y_n \in \pi_n(O_{+}^{-1}(y)) \). We say \( f^{-1} \) is transitive if for any pair of open sets \( U \) and \( V \) (non-empty) of \( X \), there exists a positive integer \( k > 0 \) such that \( f^{-k}(U) \cap V \neq \emptyset \).

One of the more commonly used definitions of chaos is that given by Devaney (2003). Suppose \( X \) is a metric space and \( f : X \to X \) is a map. Then \( f \) is chaotic on \( X \) (in the sense of Devaney) if

1. \( f \) has sensitive dependence on initial conditions;
2. \( f \) is transitive; and
3. the periodic points of \( f \) are dense in \( X \).

In this paper, when we say a map \( f : X \to X \) is chaotic, we mean \( f \) is chaotic in the sense of Devaney. Furthermore, since these these properties can be applied to \( f^{-1} \) (well-defined or not), we can use the exact same criteria for \( f^{-1} \) to be chaotic on \( X \).

Note that since \( f \) and \( f^{-1} \) share the same set of periodic points, \( f \) will have a dense set of periodic points in \( X \) if and only if \( f^{-1} \) does as well. The next theorem establishes that \( f \) is transitive if and only if \( f^{-1} \) is as well.

**Theorem 2.** Suppose \( X \) is a compact metric space, and \( f : X \to X \) is continuous. Then \( f^{-1} \) is transitive on \( X \) if and only if \( f \) is transitive on \( X \).

**Proof.** Let \( U, V \) be open sets in \( X \). Suppose there is a positive integer \( k \) such that \( f^k(U) \cap V \neq \emptyset \). But \( f^k(U) \cap V \neq \emptyset \) if and only if \( U \cap f^{-k}(V) \neq \emptyset \). So \( f^{-1} \) is transitive if and only if \( f \) is. \( \square \)

If \( X \) is a compact metric space, and \( f : X \to X \) is continuous, then the inverse limit space \( \varprojlim (X, f) \) is a compact metric space as well.\(^9\) This gives the following corollary.

**Corollary 1.** Suppose \( X \) is a compact metric space, and \( f : X \to X \) is continuous. Let \( Z = \varprojlim (X, f) \) be the inverse limit of \((X, f)\) and \( F : Z \to Z \) be the homeomorphism on \( Z \) induced by \( f \) and \( \sigma = F^{-1} \). Then \( F \) is transitive if and only if \( \sigma \) is transitive.

The only piece missing to establish that \( f \) is chaotic on \( X \) if and only if \( f^{-1} \) is chaotic on \( X \) is the property of sensitive dependence on initial conditions. This will be shown using the inverse limit space, the induced homeomorphism and its inverse (the shift map) Our argument will not establish that \( f \) has sensitive dependence on initial conditions if and only if \( f^{-1} \) has sensitive dependence on initial conditions. This in fact is not generally true. Instead, it will establish that sensitive dependence on initial conditions is an iff property.

\(^9\)See Ingram (2000), Theorem 1.3 on page 9, or Nadler (1992), Theorem 2.4 on page 19.
conditional on having a dense set of periodic points and being transitive. The structure of the argument is to show that \( f \) is chaotic on \( X \) is chaotic iff the shift map \( \sigma \) is chaotic on the inverse limits space \( \varprojlim(X,f) \). Then we show that \( \sigma \) has sensitive dependence on initial conditions iff \( f^{-1} \) does as well.

The next theorem from Banks et al. (1992) shows that sensitive dependence on initial conditions is a redundant criterion for being chaotic if the map \( f \) is continuous and the space \( X \) is a metric space.

**Theorem 3 (Banks et al. (1992)).** If \( X \) is a metric space, and \( f : X \to X \) is transitive and also has a dense set of periodic points, then \( f \) has sensitive dependence on initial conditions.

If more structure is assumed on the space \( X \), even the property of having a dense set of orbits becomes redundant. A tree \( P \) is a compact, connected metric space that can be written as the union of a finite collection \( \{P_1, P_2, \ldots, P_n\} \) of arcs such that (i) \( P_i \cap P_j \) is either empty or consists of exactly one point with that point being an endpoint of both arcs \( P_i \) and \( P_j \), and (ii) \( P \) contains no simple closed curve. (Note that \( P = \bigcup_{i=1}^n P_i \).) An interval is therefore a very simple example of a tree.

**Theorem 4 (Roe (1993)).** If \( X \) is a tree and \( f : X \to X \) is transitive, then \( f \) is chaotic.

In order to make use of this theorem, we must first establish that \( f \) has a dense set of periodic points in \( X \) iff the induced homeomorphism \( F \) has a dense set of points in the inverse limit space \( \varprojlim(X,f) \).

**Theorem 5.** Suppose \( X \) is a compact metric space and \( f : X \to X \) is continuous, \( Z := \varprojlim(X,f) \), and \( F \) denotes the homeomorphism on \( Z \) induced by the map \( f \). Then \( f \) has a dense set of periodic points in \( X \) if and only if \( F \) has a dense set of periodic points in \( Z \).

**Proof.** Suppose \( f \) has a dense set of periodic points in \( X \). Let \( u = \langle u_0, u_1, \ldots, u_n \rangle \) be a basic open set in \( Z \) and \( u \cap Z \neq \emptyset \) so \( u \cap Z \) is a non-empty open set in \( Z \). Then there exists some \( x = (x_0, x_1, x_2, \ldots) \in u \cap X \), and since, for each \( j, 0 \leq j \leq n \), \( f^j(x_n) \in u_{n-j} \), we have \( x_n \in f^{-j}(u_{n-j}) \). Thus, \( \hat{u} := \cap_{j=0}^n f^{-j}(u_{n-j}) \neq \emptyset \) and is open in \( X \). Since \( f \) has a dense set of periodic points in \( X \), there must exists a points \( y \in \hat{u} \) such that \( y \) is periodic with periodicity denoted by some positive integer \( m \). Since \( y \in u_n \) and \( f^j(y) \in u_{n-j} \) for \( j = 0, 1, \ldots, n \) we have \( \{f^n(y), f^{n-1}(y), \ldots, y, f^{m-1}(y), f^{m-1}(y), \ldots, y, \ldots\} \in u \cap X \) which is periodic of period \( m \) under \( F \).

Suppose \( F \) has a dense set of periodic in \( Z \). Let \( u_0 \) be an open set in \( X \) and \( u = \langle u \rangle \) be a basic open set in \( Z \) Since \( u \cap X \neq \emptyset \), we have \( u \cap X \) being a non-empty open set in \( Z \).
Since $F$ has a dense set of periodic points in $Z$ there exists a periodic point of say period $m$ in $u \cap Z$, call it $y = (y_0, y_1, y_2, \ldots, y_{m-1}, y_0, y_1, \ldots)$. We have $y_0 \in u_0$ being periodic (of period $m$) under $f$.

Given the result from Theorem 5, the next theorem establishes the equivalence of $f$, $F$ and $\sigma$ being chaotic.

**Theorem 6 (Chaos: $f$ iff $F$ iff $\sigma$).** Suppose $X$ is a compact metric space and $f : X \to X$ is continuous, $Z := \lim(X, f)$, $F$ denotes the homeomorphism on $Z$ induced by the map $f$, and $\sigma = F^{-1}$. Then the following are equivalent:

1. $f$ is chaotic on $X$,
2. $F$ is chaotic on $Z$,
3. $\sigma$ is chaotic on $Z$.

**Proof.** By Theorem 5, we have $f$ having a dense set of periodic points in $X$ iff $F$ has a dense set of periodic points in $Z$. By Theorem 1, we have $f$ being transitive iff on $X$ iff $F$ is transitive on $Z$. By the theorem from Banks et al. (1992), we then have $f$ chaotic iff $F$ is chaotic. Since $F$ and $\sigma$ share the same set of periodic points, $F$ will have a dense set of periodic points iff $\sigma$ has a dense set of periodic points. By Corollary 1, $F$ is transitive iff $\sigma$ is transitive. Again, by the Banks et al. (1992) theorem, we have $F$ chaotic on $Z$ iff $\sigma$ is chaotic on $Z$.

Before proving the next theorem about sensitive dependence on initial conditions for $f^{-1}$ and $\sigma$, we prove the following lemma about the metric on the factor space $X$ and the induced metric on $X^\infty$.

**Lemma 1.** Let $d$ be the metric on $X$ and $\tilde{d}$ be the metric on $X^\infty$ induced by $d$ given by

$$\tilde{d}(x, y) := \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$  

Then for $x, y \in X^\infty$, we have

1. $d(x_n, y_n) \leq \tilde{d}(\sigma^n(x), \sigma^n(y))$, and

2. Given $r > 0$, if $\tilde{d}(\sigma^k(x), \sigma^k(y)) > r$ for some $k \geq 0$, then there exists $m \geq k$ such that $d(x_m, y_m) \geq r/4$.  

15
Proof. Note that by definition of $\tilde{d}$ and the fact that $d$ is a metric, we have

$$\tilde{d}(\sigma^n(x), \sigma^n(y)) = \sum_{i=0}^{\infty} \frac{d(x_{n+i}, y_{n+i})}{2^i} \geq d(x_n, y_n).$$

Suppose $x, y \in X^\infty$, $r > 0$ and there exists $k \geq 0$ such that $\tilde{d}(\sigma^k(x), \sigma^k(y)) > r$. Suppose that $d(x_m, y_m) < r/4$ for all $m \geq k$. Then we have

$$r < \tilde{d}(\sigma^k(x), \sigma^k(y)) = \sum_{i=0}^{\infty} \frac{d(x_{k+i}, y_{k+i})}{2^i} \leq (r/4) \sum_{i=0}^{\infty} \frac{1}{2^i} = r/2,$$

which is a contradiction. \qedhere

Theorem 7. Suppose $X$ is a compact metric space and $f : X \to X$ is continuous, $Z := \lim(X, f)$, $F$ denotes the homeomorphism on $Z$ induced by the map $f$, and $\sigma = F^{-1}$. Then $\sigma$ has sensitive dependence on initial conditions iff $f^{-1}$ has sensitive dependence on initial conditions.

Proof. Suppose $f^{-1}$ has sensitive dependence on initial conditions. Then there exists $\delta > 0$ such that for each $x = (x_0, x_1, \ldots) \in Z$ and neighborhood $N$ of $x_0$, there exists $y_0 \in N$ and integer $n \geq 0$ such that $d(x_n, y_n) > \delta$. Let $x \in Z$ and $u = < u_0, u_1, \ldots, u_n >$ be a basic open set in $X^\infty$ with $x \in u \cap Z$. Then $x_i \in u_i$ for $i = 0, 1, \ldots, n$ implies $f^i(x_n) \in u_{n-i}$ for $i = 0, 1, \ldots, n$. Let $\tilde{u} := \cap_{i=0}^{n} f^{-i}(u_{n-i})$. Since $f$ is continuous and $x_n \in f^{-i}(u_{n-i})$ for $i = 0, 1, \ldots, n$, we have $\tilde{u} \neq \emptyset$ and open in $X$. Since $f^n : X \to X$ is continuous and onto, if $\theta$ is open in $X$ and $f^n(\tilde{u}) \in \emptyset$, then there exists $V \subset X$ open such that $z \in V$ and $f^n(V) \subset \emptyset$. Suppose $N$ is an open neighborhood of $x_0$. Then since $x_0 = f^n(x_n)$, there exists an open neighborhood $V \subset X$ of $x_0$ with $f^n(V) \subset N$. Consider $\tilde{u} \cap V$. This set is open and non-empty ($x_n \in \tilde{u} \cap V$) and $f^n(\tilde{u} \cap V) \subset N$. Let $w = \sigma^n(x) \equiv (x_n, x_{n+1}, \ldots)$. Since $x_n \in \tilde{u} \cap V$ and $f^{-1}$ has sensitive dependence on initial conditions there exists $y = (y_0, y_1, \ldots)$ and $m > 0$ with $y_0 \in \tilde{u} \cap V$ such that $d(w_n, y_n) = d(x_{n+m}, y_n) > \delta$. Let $p = f^n(y)$ (or equivalently $\sigma^n(p) = y$). Note $p \in u \cap Z$ since $p_n = y_0 \in \tilde{u} \cap V$ implies $p_n \in u_n, p_{n-1} = f(p_n) \in u_{n-1}, \ldots p_0 = f^n(p_n) \in u_0$. Then we have

$$\delta < d(x_{n+m}, y_m) = d(w_m, y_m) \leq \tilde{d}(\sigma^m(w), \sigma^m(y)) = \tilde{d}(\sigma^{m+n}(x), \sigma^{m+n}(p)),$$

which implies that $\sigma$ had sensitive dependence on initial conditions.

Suppose $\sigma$ has sensitive dependence on initial conditions with sensitivity constant $r > 0$. We are going to show that $r/4 > 0$ is a sensitivity constant for $f^{-1}$. Let $x_0 \in X$, $u_0$ an open neighbor hoo of $x_0$, and $x = (x_0, x_1, \ldots) \in Z$. Then $x \in u := < u_0 > \cap Z$ is open.
and non-empty. Since $\sigma$ has sensitive dependence on initial conditions, there exits $y \in u$ and $k \geq 0$ such that $d(\sigma^k(x), \sigma^k(y)) > r$. Note that $y_0 \in u_0$. By Lemma 1, there exists $m \geq k$ such that $d(x_m, y_m) > r/4$, which implies $f^{-1}$ has sensitive dependence on initial conditions. 

The previous theorem leads to the next theorem establishing that $\sigma$ is chaotic iff $f^{-1}$ is chaotic.

Theorem 8. Suppose $X$ is a compact metric space and $f : X \to X$ is continuous, $Z = \operatorname{lim}(X,f)$, $F$ denotes the homeomorphism on $Z$ induced by the map $f$, and $\sigma = F^{-1}$. Then $\sigma$ is chaotic on $Z$ iff $f^{-1}$ is chaotic on $X$.

Proof. The properties of a dense set of periodic points and transitivity are if and only if properties for $f^{-1}$, $f$, $F$ and $\sigma$. By Theorem 7, $f^{-1}$ had sensitive dependence on initial conditions if and only if $\sigma$ does. 

Our main result in the next theorem then follows directly from Theorems 6 and 8.

Theorem 9. Suppose $X$ is a compact metric space and $f : X \to X$ is continuous. Then $f$ is chaotic iff $f^{-1}$ is chaotic.

4 Conclusion

In some economic models like the OLG or cash-in-advance model of money, the dynamical system characterizing equilibria in the model has ill-defined forward dynamics but well-defined backward dynamics. In this paper, we have offered a definition of chaotic for such a dynamical system that is in the spirit of Devaney. Furthermore, by utilizing the inverse limit space, we have been able to prove that for such dynamical systems there is chaos going forward in time if and only if there is chaos going backward in time. Our results are very general and apply whenever the backward map is continuous and the state space is a compact metric space.
References


